Problem Set — Week 3, Session 2 · Solutions

Mathematical Statistics I · Wackerly §2.8 · ADA University

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260926

A retail bank's customers are surveyed. Let A be the event that a customer uses the mobile app and B the event that the customer holds a credit card:

$$P(A) = 0.42, \qquad P(B) = 0.45, \qquad P(A \cap B) = 0.32.$$

Give every probability to at least four significant figures.

(a) $P(A \cup B) =$ ____________

(b) $P(\bar{A} \cap \bar{B}) =$ ____________

(c) $P(A \cap \bar{B}) =$ ____________

Solution

(a) The additive law, Theorem 2.6:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.42 + 0.45 - 0.32 = 0.5500.$$

Adding $P(A)$ and $P(B)$ counts the customers in both events twice, and subtracting the intersection once repairs exactly that.

(b) By De Morgan's law $\bar{A} \cap \bar{B} = \overline{A \cup B}$, so Theorem 2.7 gives $1 - 0.5500 = 0.4500$.

(c) A splits into the part inside B and the part outside it, and these are mutually exclusive, so $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.42 - 0.32 = 0.1000$.

A check worth doing on any problem of this shape: the four pieces $A \cap B$, $A \cap \bar{B}$, $\bar{A} \cap B$, $\bar{A} \cap \bar{B}$ partition the sample space, so their probabilities must sum to 1. Here $0.32 + 0.1000 +$ $P(\bar{A} \cap B)$ $+ 0.4500 = 1$, which fixes the one piece not asked for.

Problem 2

seed 20260926

A construction firm bids for public tenders. Let A be the event that a bid is filed before the deadline and B the event that the bid wins. Historically

$$P(A) = 0.43, \qquad P(B \mid A) = 0.57, \qquad P(B) = 0.4900.$$

Give every probability to at least four significant figures.

(a) $P(A \cap B) =$ ______________

(b) $P(A \cup B) =$ ______________

(c) $P(A \mid B) =$ ______________

Solution

(a) Theorem 2.5, the multiplicative law:

$$P(A \cap B) = P(A)\,P(B \mid A) = 0.43 \times 0.57 = 0.2451.$$

(b) Theorem 2.6 now has everything it needs: $P(A \cup B) = 0.43 + 0.4900 - 0.2451 = 0.6749$.

(c) The multiplicative law can be read from either end, and the second reading is $P(A \cap B) = P(B)P(A \mid B)$, so

$$P(A \mid B) = \frac{0.2451}{0.4900} = 0.5002.$$

The two laws divide the work between them cleanly: the multiplicative law builds intersections out of conditional probabilities, and the additive law turns intersections into unions. Almost every calculation in Section 2.9 is a sequence of those two moves, chosen so that the quantities on the right are the ones you were actually given.

Problem 3

seed 20260926

An operations manual records that a particular machine injures an operator on about 1 shift in 42. Shifts are independent of one another.

Give every probability to at least four significant figures, and enter them as decimals.

(a) $P(\text{an injury on a single shift}) =$ ______________

(b) $P(\text{at least one injury in 8 shifts}) =$ ______________

(c) A colleague argues that since the chance is 1 in 42 per shift, an operator working 42 shifts is certain to be injured. Compute the correct probability of at least one injury in 42 shifts. ______________

Solution

(a) $p = 1/42 = 0.023810$.

(b) Go through the complement. Independence makes a run of clean shifts a product, so the probability of no injury in 8 shifts is

$$0.976190^8 = 0.824663,$$

and Theorem 2.7 gives $1 - 0.824663 = 0.175337$.

Attacking this directly would mean the probability of a union of 8 events, which by inclusion–exclusion has $2^8 - 1$ terms. The complement has one. Whenever a question says "at least one", the complement is the first thing to try.

(c) The colleague's reasoning adds probabilities that are not mutually exclusive — nothing prevents two injuries in 42 shifts, and the additive law subtracts exactly the overlap that the argument ignores. The correct value is

$$1 - 0.976190^{42} = 0.636544,$$

which is high but not 1, and never reaches 1 for any finite number of shifts. As m grows, $(1 - 1/m)^m \to e^{-1} \approx 0.3679$, so this probability settles near $1 - 0.3679 = 0.6321$ and stays there. Doing something with one-in-m odds m times leaves you roughly a 37% chance of never seeing it at all.

Problem 4

seed 20260926

Two events are defined on the firms in a supervisory register: A, the firm filed its accounts on time, and B, the firm has no overdue tax. It is known that

$$P(A) = 0.71 \qquad \text{and} \qquad P(B) = 0.75,$$

and nothing else. Give every probability to at least four significant figures.

(a) What is the smallest value $P(A \cap B)$ could take? ____________

(b) What is the largest value $P(A \cap B)$ could take? ____________

(c) Could $P(A \cap B) = 0.77$?

- ○ Impossible: too small
- ○ Possible
- ○ Impossible: too large

Solution

(a) The additive law rearranges to $P(A \cap B) = P(A) + P(B) - P(A \cup B)$, and no probability exceeds 1, so $P(A \cup B) \leq 1$ forces

$$P(A \cap B) \geq 0.71 + 0.75 - 1 = 0.4600.$$

The bound is attained when $A \cup B$ is the whole register: the two events overlap as little as they can, and since their probabilities already sum to more than 1 they cannot avoid each other entirely.

(b) $A \cap B \subset A$ and $A \cap B \subset B$, so $P(A \cap B) \leq \min\{0.71, 0.75\} = 0.7100$, attained when the smaller event sits entirely inside the larger.

(c) The value 0.77 is above the upper bound [`[0.7100]`], and an intersection cannot be more probable than either of the events containing it.

The general statement is worth carrying:

$$\max\{0,\ P(A)+P(B)-1\}\ \leq\ P(A\cap B)\ \leq\ \min\{P(A),P(B)\}.$$

Both halves are used as sanity checks on reported figures. If a survey reports that 85% of firms file on time, 80% have no overdue tax, and 55% do both, the three numbers are mutually impossible, and that can be seen without any further information about the firms.

Problem 5

seed 20260926

An invoice is settled early (E), on its due date (D), late (L), or not at all (N). Exactly one of the four happens, and

$$P(E)=0.1900,\qquad P(D)=0.4000,\qquad P(L)=0.2300.$$

Give every probability to at least four significant figures.

(a) $P(N)=$ ______________

(b) $P(E\cup D)=$ ______________

(c) $P(\bar{L})=$ ______________

(d) $P(D\cup L)=$ ______________

Solution

The four outcomes are mutually exclusive and one of them must occur, so their probabilities sum to 1 and every union among them is a plain sum — the last term of the additive law is zero in each case.

(a) $P(N)=1-0.1900-0.4000-0.2300=0.1800$.

(b) $P(E\cup D)=0.1900+0.4000=0.5900$, with no correction, because an invoice cannot be settled both early and on the due date.

(c) Theorem 2.7: $P(\bar{L})=1-0.2300=0.7700$.

(d) $P(D\cup L)=0.4000+0.2300=0.6300$.

The pattern here is the sample-point method of Section 2.4 wearing different clothes. When the events in question are the individual outcomes of a partition, probabilities simply add, and Theorem 2.6 has nothing to correct. The additive law earns its keep only when events can happen together.

Problem 6

seed 20260926

A telecommunications regulator receives complaints about three faults: slow speed (A), dropped connections (B) and billing errors (C). For a subscriber drawn at random from the complaints register,

$$P(A) = 0.4000, \quad P(B) = 0.3500, \quad P(C) = 0.4100,$$

$$P(A \cap B) = 0.1300, \quad P(A \cap C) = 0.1400, \quad P(B \cap C) = 0.1300, \quad P(A \cap B \cap C) = 0.0500.$$

Give every probability to at least four significant figures.

(a) $P(A \cup B \cup C) =$ ______________

(b) $P(\text{none of the three faults}) =$ ______________

(c) $P(\text{slow speed and neither of the other two}) =$ ______________

Solution

(a) Applying Theorem 2.6 twice gives the three-event form:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Substituting, $0.4000 + 0.3500 + 0.4100 - 0.1300 - 0.1400 - 0.1300 + 0.0500 = 0.8100$.

The alternating signs are a counting argument, not a formula to memorise. A subscriber in exactly two of the events is added twice and subtracted once, so is counted once. A subscriber in all three is added three times, subtracted three times, and would vanish — which is why the triple intersection is added back.

(b) Theorem 2.7 with De Morgan: $P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - 0.8100 = 0.1900$.

(c) Strip the overlaps off A. The subscribers in A and at least one other event have probability $P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$, so

$$P(A \cap \bar{B} \cap \bar{C}) = 0.4000 - 0.1300 - 0.1400 + 0.0500 = 0.1800.$$

The triple intersection reappears for the same reason as before: subtracting both pairwise overlaps removes the subscribers who have all three faults twice over.

Problem 7

seed 20260926

A tender passes through three stages in order: a technical screen (T), a financial screen (F), and the award decision (W). A bid reaches a stage only if it has passed the previous one. For a bid drawn at random,

$$P(T) = 0.66, \qquad P(F \mid T) = 0.65, \qquad P(W \mid T \cap F) = 0.71.$$

Give every probability to at least four significant figures.

(a) $P(T \cap F) =$ ______________

(b) $P(T \cap F \cap W) =$ ______________

(c) $P($the bid is not awarded$) =$ ______________

(d) $P($the bid passes the technical screen but fails the financial one$) =$ ______________

Solution

(a) Theorem 2.5: $P(T \cap F) = P(T)P(F \mid T) = 0.66 \times 0.65 = 0.4290$.

(b) Apply it again, taking $T \cap F$ as the first event:

$$P(T \cap F \cap W) = P(T)\,P(F \mid T)\,P(W \mid T \cap F) = 0.66 \times 0.65 \times 0.71 = 0.3046.$$

Each conditioning event is everything that has happened so far, not merely the immediately preceding stage. That is what makes the chain valid without any independence assumption — and it is exactly the point where a careless $P(W \mid F)$ would change the meaning.

(c) A bid is awarded only by passing all three stages, so the complement: $1 - 0.3046 = 0.6954$.

(d) $P(T \cap \bar{F}) = P(T)\,P(\bar{F} \mid T) = 0.66 \times (1 - 0.65) = 0.2310$.

Conditional probabilities obey Theorem 2.7 in their second argument as well: $P(\bar{F} \mid T) = 1 - P(F \mid T)$, because $P(\cdot \mid T)$ is itself a probability.

Notice the arithmetic of a pipeline. Three stages that individually look generous compose to 0.3046, and adding a fourth stage can only lower it. Firms that complain about tender success rates are often describing nothing more than the product of several honest screens.

Problem 8

seed 20260926

Three risks are monitored at a plant: a power interruption (A_1), a raw material shortage (A_2) and a labour stoppage (A_3). A shortage and a stoppage never occur in the same month, so $P(A_2 \cap A_3) = 0$, while

$$P(A_1) = 0.3200, \quad P(A_2) = 0.26, \quad P(A_3) = 0.27,$$

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = 0.06.$$

Give every probability to at least four significant figures.

(a) $P(A_1 \cap A_2 \cap A_3) =$ ______________

(b) $P($at least one of the three risks occurs$) =$ ______________

(c) $P($none of the three occurs$) =$ ______________

(d) By how much does $P(A_1) + P(A_2) + P(A_3)$ exceed the answer to (b)? ______________

Solution

(a) $A_1 \cap A_2 \cap A_3 \subset A_2 \cap A_3$, and a subset of an event of probability zero has probability zero, so the triple intersection is 0.

(b) Put that into the three-event additive law. Two of its correction terms vanish — $P(A_2 \cap A_3) = 0$ and the triple intersection — and the two remaining pairwise terms are equal, so

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - 2P(A_1 \cap A_2),$$

which is $0.3200 + 0.26 + 0.27 - 2(0.06) = 0.7300$.

(c) $1 - 0.7300 = 0.2700$.

(d) The excess is exactly the $2P(A_1 \cap A_2) = 0.1200$ that had to be subtracted.

That excess has a plain reading, and it is the reason risk registers overstate exposure. Adding the three probabilities counts every month in which a power interruption coincides with one of the other two risks twice. Summing risk probabilities is safe only when the risks are mutually exclusive; otherwise the sum is an upper bound, sometimes a very loose one, and here it overstates the true figure by 0.1200.

Problem 9

seed 20260926

A survey of households reports that 0.37 subscribe to fixed broadband (A), 0.40 subscribe to pay television (B), and 0.6920 subscribe to at least one of the two.

Give every probability to at least four significant figures.

(a) $P(A \cap B) =$ ______________

(b) $P(A \mid B) =$ ______________

(c) Are A and B independent?

- ◯ Independent
- ◯ Dependent

(d) If instead the two subscriptions were independent, what would $P(A \cup B)$ be? ______________

Solution

(a) Rearranging Theorem 2.6,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.37 + 0.40 - 0.6920 = 0.0780.$$

This is how the additive law is used most often in practice. Unions are what surveys measure — “at least one service” is a question a respondent can answer — and intersections are what the analysis needs.

(b) $P(A \mid B) = 0.0780/0.40 = 0.1950$.

(c) Independence would require $P(A \cap B) = P(A)P(B) = 0.1480$. The observed intersection is 0.0780, so the events are Dependent.

(d) Under independence the additive law would give $0.37 + 0.40 - 0.1480 = 0.6220$.

Comparing this with the reported 0.6920 is a quick diagnostic: whenever the two services are bought together more often than independence predicts, the union is **smaller** than the independent benchmark, because the same households are being counted in both services. Bundling makes take-up look broader than it is.

Problem 10

seed 20260926

In a sample of manufacturing firms, A is the event that a firm exports to Turkey and B the event that it exports to Russia, with

$$P(A) = 0.32, \qquad P(B) = 0.36, \qquad P(A \cap B) = 0.24.$$

Give every probability to at least four significant figures.

(a) $P(\overline{A \cup B}) =$ ______________

(b) $P(\text{the firm exports to exactly one of the two}) =$ ______________

(c) $P(\bar{A} \cap B) =$ ______________

Solution

First the union: $P(A \cup B) = 0.32 + 0.36 - 0.24 = 0.4400$.

(a) Theorem 2.7 gives $1 - 0.4400 = 0.5600$. By De Morgan's law the same number is $P(\bar{A} \cap \bar{B})$ — the firms exporting to neither.

(b) "Exactly one" is "at least one" with "both" removed, and $A \cap B \subset A \cup B$, so the probabilities subtract:

$$P(\text{exactly one}) = 0.4400 - 0.24 = 0.2000.$$

(c) B splits into the firms that also export to Turkey and those that do not: $P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0.36 - 0.24 = 0.1200$.

It is worth laying the four disjoint pieces out once, because every question of this kind is answered by picking pieces off the list:

$$0.24 \text{ (both)}, \quad 0.1200 \text{ (Russia only)}, \quad \text{Turkey only}, \quad 0.5600 \text{ (neither)},$$

and these four must sum to 1. Once the table is written down, no further use of Theorem 2.6 is needed — the additive law's real job was to produce it.

Problem 11

seed 20260926

Three suppliers deliver to a plant. Their deliveries are late independently of one another, with probabilities

$$P(L_1) = 0.18, \qquad P(L_2) = 0.23, \qquad P(L_3) = 0.26.$$

Give every probability to at least four significant figures.

(a) $P(L_1 \cap L_2) =$ ____________

(b) $P(L_1 \cup L_2) =$ ____________

(c) $P(L_1 \cup L_2 \cup L_3) =$ ____________

(d) $P(L_1 \cap L_2 \cap L_3) =$ ____________

Solution

(a) Independence turns Theorem 2.5 into a plain product: $0.18 \times 0.23 = 0.0414$.

(b) Theorem 2.6 then gives $0.18 + 0.23 - 0.0414 = 0.3686$.

(c) Here the two laws part company on effort. Through the complement,

$$P(\textbf{no supplier is late}) = 0.8200 \times 0.7700 \times 0.7400 = 0.46724,$$

so the union is $1 - 0.46724 = 0.5328$. The inclusion–exclusion route needs seven terms and returns the same number; with ten suppliers it would need 1023, and the complement would still need one product.

(d) $0.18 \times 0.23 \times 0.26 = 0.01076$.

Compare (c) and (d). At least one supplier is late in about 0.5328 of months, and all three are late together in 0.01076 — a difference of orders of magnitude. A plant that can tolerate one late supplier and a plant that needs all three on time are facing very different risks from the same three numbers, which is why “the risk” is not a well-posed phrase until the event has been named.

Problem 12

seed 20260926

For two events defined on a portfolio it is known only that

$$P(A) = 0.27 \qquad \text{and} \qquad P(B) = 0.30.$$

Give every probability to at least four significant figures.

(a) The smallest possible value of $P(A \cap B)$ is ____________

(b) The largest possible value of $P(A \cap B)$ is ____________

(c) The smallest possible value of $P(A \cup B)$ is ____________

(d) The largest possible value of $P(A \cup B)$ is ______________

Solution

Because $P(A) + P(B) = 0.5700 < 1$, the two events have room to avoid each other completely, and that is what makes this case different from the one in which the marginals sum to more than 1.

(a) The intersection can be empty: place A and B in disjoint parts of the sample space, which is possible precisely because their probabilities leave $1 - 0.5700$ of it unused. So the smallest value is 0.

(b) $A \cap B$ is contained in both events, so it cannot exceed $\min\{0.27, 0.30\} = 0.2700$, attained when the smaller event lies inside the larger.

(c) and **(d)** follow from the additive law, since $P(A \cup B) = 0.5700 - P(A \cap B)$ moves in the opposite direction to the intersection. The largest intersection gives the smallest union, $0.5700 - 0.2700 = 0.3000 = \max\{P(A), P(B)\}$; the empty intersection gives the largest union, 0.5700.

The general statement, which covers this problem and the case $P(A) + P(B) > 1$ alike:

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq \min\{1,\ P(A) + P(B)\}.$$

The upper bound is the union bound, and it is used constantly in probability precisely because it needs nothing at all about how the events relate. The lower bound says the union is at least as likely as either event on its own, which is obvious once stated and still worth stating: adding an event to a union can never make it less likely.

Problem 13

seed 20260926

Two inspectors examine the same consignment independently of each other's paperwork, but they are looking at the same goods, so their verdicts are not independent events. Let A and B be the events that inspector 1 and inspector 2 respectively flag the consignment. It is known that

$$P(A) = 0.23, \qquad P(B \mid A) = 0.67, \qquad P(B \mid \bar{A}) = 0.21.$$

Give every probability to at least four significant figures.

(a) $P(A \cap B) =$ ______________

(b) $P(B) =$ ______________

(c) $P(A \cup B) =$ ______________

(d) $P(\text{neither inspector flags it}) =$ ______________

Solution

(a) Theorem 2.5: $P(A \cap B) = 0.23 \times 0.67 = 0.1541$.

(b) Split B along A. The pieces $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive and make up B, and each is built by the multiplicative law:

$$P(B) = P(A)P(B \mid A) + P(\bar{A})P(B \mid \bar{A}) = 0.23 \times 0.67 + 0.7700 \times 0.21 = 0.3158.$$

(c) Theorem 2.6: $0.23 + 0.3158 - 0.1541 = 0.3917$.

(d) $1 - 0.3917 = 0.6083$.

Two things to take from this. The calculation in (b) is the multiplicative and additive laws used together — the pattern that Section 2.10 will name the law of total probability. Getting it from the two theorems first is worth more than memorising it later.

And the dependence is real: $P(A)P(B) = 0.0726$ against an actual intersection of 0.1541. Treating the two inspections as independent would misstate the chance that both flag the same consignment, which is precisely the figure a quality system is built on. Two inspectors looking at the same goods agree more often than chance — that is what makes them inspectors, and what makes the independence shortcut wrong here.

Problem 14

seed 20260926

A municipal survey of small shops finds that 0.39 accept card payments (A), 0.7600 accept cards or issue electronic receipts or both $(A \cup B)$, and 0.16 do both $(A \cap B)$.

Give every probability to at least four significant figures.

(a) $P(B) =$ ______________

(b) $P(\text{the shop does neither}) =$ ______________

(c) $P(B \mid A) =$ ______________

(d) $P(A \cap \bar{B}) =$ ______________

Solution

(a) Theorem 2.6 has four quantities in it and any three determine the fourth:

$$P(B) = P(A \cup B) + P(A \cap B) - P(A) = 0.7600 + 0.16 - 0.39 = 0.5300.$$

(b) $1 - 0.7600 = 0.2400$.

(c) Definition 2.9: $P(B \mid A) = 0.16/0.39 = 0.4103$. Among shops that already take cards, this fraction also issue electronic receipts.

(d) $P(A) - P(A \cap B) = 0.39 - 0.16 = 0.2300$ — the shops that take cards but still write receipts by hand, which for a tax authority planning an e-receipt rollout is the group that matters.

The moral of (a) is worth stating once. The additive law is an identity among four numbers, not a recipe for computing one particular one. In a survey the easiest quantity to measure is often the union, because “do you do either?” is a simpler question than “do you do both?”; the identity is then run backwards, exactly as here.

Problem 15

seed 20260926

A batch of 35 castings contains 7 that are defective. An inspector draws 5 of them at random without replacement.

Give every probability to at least four significant figures.

(a) $P(\text{none of the 5 drawn is defective}) =$ ____________

(b) $P(\text{at least one of the 5 drawn is defective}) =$ ____________

(c) Suppose instead each casting were returned to the batch before the next draw. What would the probability of at least one defective be then? ____________

Solution

(a) Chain the multiplicative law over the 5 draws. Each factor is the conditional probability that the next casting is sound given that all the earlier ones were:

$$P = \frac{28}{35} \cdot \frac{27}{34} \cdot \frac{26}{33} \cdots = 0.302743,$$

the product running over all 5 draws: each step removes one sound casting from the numerator and one casting from the denominator.

(b) Theorem 2.7: $1 - 0.302743 = 0.697257$. The direct route would be a union of 5 events that are neither mutually exclusive nor independent, and the inclusion–exclusion expansion is far more work for the same number.

(c) With replacement the draws become independent and every factor is the same, 0.800000, so the probability of no defective is $0.800000^5 = 0.327680$ and the answer is 0.672320.

The two answers are close but not equal, and the direction of the difference is worth knowing. Drawing without replacement makes a second defective slightly **less** likely once one has been found, so the sampling distribution is a little tighter than the independent calculation suggests. The gap shrinks as the batch grows: for a batch of several thousand, the independence approximation is harmless, which is why acceptance sampling in practice uses it.

Problem 16

seed 20260926

An exploration company drills wells that strike gas independently of one another, each with probability 0.120. The board wants the probability of at least one strike to be at least 0.85.

(a) What is the smallest number of wells that achieves this? Enter a whole number. ____________

Give the remaining probabilities to at least four significant figures.

(b) With that many wells, $P(\text{at least one strike}) =$ ______________

(c) With one well fewer, $P(\text{at least one strike}) =$ ______________

Solution

(a) With n independent wells, no strike has probability 0.8800^n, so the requirement is

$$1 - 0.8800^{\,n} \geq 0.85, \qquad \text{that is} \qquad 0.8800^{\,n} \leq 1 - 0.85.$$

Taking logarithms and remembering that $\ln 0.8800$ is **negative**, so the inequality turns around:

$$n \;\geq\; \frac{\ln(1 - 0.85)}{\ln 0.8800} = 14.8406,$$

and since n must be a whole number, $n = 15$.

The sign of the logarithm is where this calculation is usually got wrong. Dividing an inequality by a negative number reverses it, and here that is the difference between a floor and a ceiling.

(b) $1 - 0.8800^{15} = 0.8530$, which clears 0.85.

(c) $1 - 0.8800^{14} = 0.8330$, which does not.

Two cautions about what this number means. It assumes the wells are independent, and wells drilled into the same geological structure are not — the real programme is closer to one well repeated than to 15 independent trials, and the true probability of at least one strike is then lower, often much lower. And “at least one strike” is not the same as a commercially viable programme: 15 dry wells cost 15 times one dry well, so the board’s criterion should also weigh the cost of the tail in which the one strike arrives last.

Problem 17

seed 20260926

A bank’s operational risk register lists three independent failure events for the coming quarter — a payments outage, a reconciliation error and a third-party breach — with probabilities

$$0.13, \qquad 0.16, \qquad 0.18.$$

Give every probability to at least four significant figures.

(a) The register reports “total operational risk” by adding the three figures. What does it report? ______________

(b) What is the correct probability that at least one of the three occurs? ______________

(c) By how much does the reported figure overstate the correct one? ______________

Solution

(a) The reported figure is the plain sum, $0.13 + 0.16 + 0.18 = 0.4700$.

(b) The events are independent, so the complement is a product:

$$P(\text{at least one}) = 1 - (0.8700)(0.8400)(0.8200) = 0.4007.$$

(c) The overstatement is $0.4700 - 0.4007 = 0.0693$.

The sum is the **union bound**: for any events whatever,

$$P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3),$$

with equality only when no two of them can occur together. It follows from the additive law by dropping the subtracted intersections, which are non-negative. The bound is genuinely useful — it needs no information at all about how the events relate, which is why it survives into far more advanced work — but it is a bound, not a value.

Here the gap of 0.0693 is modest because the three probabilities are small and the double-counted region is therefore small. A register of twenty such items would overstate considerably more, and one made of items with probabilities near 1 could report a "total risk" above 1, which is the point at which the practice announces its own error.

Problem 18

seed 20260926

A bank holds loans to two firms in the same supply chain. Over the coming year firm 1 defaults with probability 0.07 and firm 2 with probability 0.08. The two are not independent: if firm 1 defaults, firm 2's default probability is 4.5 times its unconditional value.

Give every probability to at least four significant figures.

(a) $P(D_2 \mid D_1) =$ ______________

(b) $P(D_1 \cap D_2) =$ ______________

(c) $P(D_1 \cup D_2) =$ ______________

(d) $P(\text{neither firm defaults}) =$ ______________

Solution

(a) $P(D_2 \mid D_1) = 4.5 \times 0.08 = 0.3600$.

(b) Theorem 2.5: $P(D_1 \cap D_2) = P(D_1)P(D_2 \mid D_1) = 0.07 \times 0.3600 = 0.02520$.

(c) Theorem 2.6: $0.07 + 0.08 - 0.02520 = 0.1248$.

(d) $1 - 0.1248 = 0.8752$.

Now compare with what an independence assumption would have produced. It would put the joint default at $0.07 \times 0.08 = 0.00560$ instead of 0.02520 — understating the event the bank most needs to price, by a factor of 4.5 — while the union would come out at 0.1444, barely different from 0.1248.

That asymmetry is the whole lesson. Dependence between rare events hardly moves the probability that **something** goes wrong, and moves the probability that **everything** goes wrong enormously. A portfolio measured by the first number looks almost unaffected by correlation; the same portfolio measured by the second does not. Which number a risk report shows is therefore not a presentational choice.

Problem 19

seed 20260926

Among a telecoms operator's customers, let C be the event that a customer is on a fixed tariff, A that the customer complains about speed and B that the customer complains about billing. The operator reports

$$P(C) = 0.42, \quad P(A \mid C) = 0.45, \quad P(B \mid C) = 0.49, \quad P(A \cap B \mid C) = 0.28.$$

Give every probability to at least four significant figures.

(a) $P(A \cup B \mid C) =$ ____________

(b) $P(\bar{A} \cap \bar{B} \mid C) =$ ____________

(c) $P(A \cap B \cap C) =$ ____________

(d) $P(A \cap \bar{B} \mid C) =$ ____________

Solution

Fix the conditioning event C and consider the assignment $Q(\cdot) = P(\cdot \mid C)$. It is non-negative, $Q(S) = 1$, and it adds over mutually exclusive events — so Q satisfies the three axioms of Section 2.3 and **is** a probability. Every theorem of this chapter therefore applies to it unchanged, which is all parts (a), (b) and (d) need.

(a) The additive law for Q: $0.45 + 0.49 - 0.28 = 0.6600$.

(b) Theorem 2.7 for Q, with De Morgan: $1 - 0.6600 = 0.3400$.

(c) This one leaves the conditional world, so the multiplicative law is needed: $P(A \cap B \cap C) = P(C)\, P(A \cap B \mid C) = 0.42 \times 0.28 = 0.1176$.

(d) Again inside Q: $Q(A) - Q(A \cap B) = 0.45 - 0.28 = 0.1700$.

The distinction between (c) and the rest is the one to hold on to. Probabilities conditioned on the **same** event combine by the ordinary rules; the moment the conditioning event changes, or is dropped, the multiplicative law is required to move between the two scales. Reports that mix conditional and unconditional percentages in one table — "45% of fixed-tariff customers and 12% of all customers" — invite exactly the error of adding them.

Problem 20

seed 20260926

Every shipment arriving at a border post is selected for physical inspection with probability 0.28. Let I be the event that a shipment is inspected and F the event that a fault in its declaration is eventually found. It is known that

$$P(F \mid I) = 0.75 \qquad \text{and} \qquad P(F \mid \bar{I}) = 0.21.$$

Give every probability to at least four significant figures.

(a) $P(I \cap F) =$ ____________

(b) $P(F) =$ ____________

(c) $P(\bar{F}) =$ ____________

(d) $P(I \mid F) =$ ____________

Solution

(a) Theorem 2.5: $P(I \cap F) = P(I)P(F \mid I) = 0.28 \times 0.75 = 0.2100$.

(b) F is the union of the two mutually exclusive events $I \cap F$ and $\bar{I} \cap F$, so their probabilities add, and each is built by the multiplicative law:

$$P(F) = 0.28 \times 0.75 + 0.7200 \times 0.21 = 0.2100 + 0.1512 = 0.3612.$$

(c) Theorem 2.7: $1 - 0.3612 = 0.6388$.

(d) Read the multiplicative law from the other end:

$$P(I \mid F) = \frac{P(I \cap F)}{P(F)} = \frac{0.2100}{0.3612} = 0.5814.$$

Parts (b) and (d) together are the whole content of Section 2.10 — the law of total probability and Bayes' rule — assembled here out of nothing but Theorems 2.5 and 2.6. It is worth doing once in this form, because the two named results are otherwise easy to treat as formulas to be looked up rather than as two applications of laws you already have.

The border post's own figure is (d): of the faults that come to light, 0.5814 were caught at the post rather than surfacing later. Raising the inspection rate $P(I)$ raises that share, and the sensible question is whether the faults caught are worth the shipments delayed — a question the arithmetic frames but does not answer.

Answer key

Problem	Answers
1	(a) 0.55, (b) 0.45, (c) 0.1
2	(a) 0.2451, (b) 0.6749, (c) 0.500204

3	(a) 0.0238095, (b) 0.175337, (c) 0.636544
4	(a) 0.46, (b) 0.71, (c) Impossible: too large
5	(a) 0.18, (b) 0.59, (c) 0.77, (d) 0.63
6	(a) 0.81, (b) 0.19, (c) 0.18
7	(a) 0.429, (b) 0.30459, (c) 0.69541, (d) 0.231
8	(a) 0, (b) 0.73, (c) 0.27, (d) 0.12
9	(a) 0.078, (b) 0.195, (c) Dependent, (d) 0.622
10	(a) 0.56, (b) 0.2, (c) 0.12
11	(a) 0.0414, (b) 0.3686, (c) 0.532764, (d) 0.010764
12	(a) 0, (b) 0.27, (c) 0.3, (d) 0.57
13	(a) 0.1541, (b) 0.3158, (c) 0.3917, (d) 0.6083
14	(a) 0.53, (b) 0.24, (c) 0.410256, (d) 0.23
15	(a) 0.302743, (b) 0.697257, (c) 0.67232
16	(a) 15, (b) 0.853026, (c) 0.832984
17	(a) 0.47, (b) 0.400744, (c) 0.069256
18	(a) 0.36, (b) 0.0252, (c) 0.1248, (d) 0.8752
19	(a) 0.66, (b) 0.34, (c) 0.1176, (d) 0.17
20	(a) 0.21, (b) 0.3612, (c) 0.6388, (d) 0.581395