Week 4, Problem Set 1 - worked solutions

Wackerly 2.9 - the event-composition method

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A retail bank sends a pre-approved credit-card offer to all of its current account holders. Of these customers, 52% are salaried employees and the remaining 48% are self-employed. Among salaried customers 56% accept the offer; among the self-employed 18% accept.

One customer is chosen at random. Let F be the event that the customer accepts the offer, S that the customer is salaried, and E that the customer is self-employed.

Give every probability to at least four significant figures.

(a) $P(F \cap S) =$ ______________

(b) $P(F \cap E) =$ ______________

(c) $P(F) =$ ______________

(d) The probability that the customer is self-employed and declines the offer: $P(E \cap \overline{F}) =$ ______________

Every customer is either salaried or self-employed and not both, so S and E are mutually exclusive and $S \cup E$ is the whole sample space. That is what lets us write the event of interest as a composition (step 3 of the event-composition method):

$$F = (F \cap S) \cup (F \cap E),$$

a union of two mutually exclusive events.

(a) The multiplicative law (Theorem 2.5): $P(F \cap S) = P(F \mid S)P(S) = 0.56 \times 0.52 = 0.2912$.

(b) Likewise $P(F \cap E) = P(F \mid E)P(E) = 0.18 \times 0.48 = 0.0864$.

(c) The additive law (Theorem 2.6) with $P[(F \cap S) \cap (F \cap E)] = 0$:

$$P(F) = P(F \cap S) + P(F \cap E) = 0.2912 + 0.0864 = 0.3776.$$

Note that $P(F)$ lies between the two segment rates 0.18 and 0.56: it is their average weighted by the segment shares.

(d) Given E, declining is the complement of accepting, so $P(\overline{F} \mid E) = 1 - 0.18 = 0.82$, and by Theorem 2.5 $P(E \cap \overline{F}) = 0.82 \times 0.48 = 0.3936$.

Problem 2

seed 20260925

A payments company screens every card transaction with two fraud-detection engines bought from different vendors, each trained on its own data and running on its own servers, so that they operate independently. Each engine fails to flag a fraudulent transaction with probability 0.140.

A fraudulent transaction arrives.

Give every probability to at least four significant figures. You may enter an arithmetic expression such as 0.05^3 instead of a decimal.

(a) The probability that it goes undetected (no engine flags it): ______________

(b) The probability that it is flagged by both engines: ______________

(c) The probability that it is flagged by exactly one of the engines: ______________

(d) On a given day 90 fraudulent transactions arrive, and the engines treat each one independently of the others. What is the probability that at least one of the 90 goes undetected? ______________

Let M_i be the event that engine i misses the transaction, $i = 1, \ldots, 2$, so $P(M_i) = 0.140$ and $P(\overline{M_i}) = 0.860$.

(a) "Undetected" is the intersection $M_1 \cap \cdots \cap M_2$. Independence turns the multiplicative law (Theorem 2.5) into a plain product:

$$P(\text{undetected}) = 0.140^2 = 0.0196.$$

(b) Likewise $P(\overline{M_1} \cap \cdots \cap \overline{M_2}) = 0.860^2 = 0.7396$.

(c) "Exactly one engine flags it" is the union of 2 mutually exclusive events, one for each engine that could be the one that flags: $\overline{M_1} \cap M_2 \cap \cdots$, $M_1 \cap \overline{M_2} \cap \cdots$, and so on. Each has probability 0.860×0.140^1, so by the additive law

$$P(\text{exactly one}) = 2 \times 0.860 \times 0.14 = 0.2408.$$

(d) Composing again: "at least one of the 90 slips through" is the complement of "every one of the 90 is caught", and each is caught with probability $1 - 0.0196 = 0.98040000$. By Theorem 2.7 and independence across transactions,

$$P(\text{at least one undetected}) = 1 - (0.98040000)^{90} = 1 - 0.168383 = 0.8316.$$

A per-transaction miss rate that looks negligible in (a) becomes a daily figure that a risk committee has to look at, simply because the complement of "at least one" is an intersection of many events.

Problem 3

seed 20260925

A salary payment from a company's account to an employee's account at another bank passes through four stages, in order, and completes only if every stage works:

1. the payer bank's online banking gateway accepts the instruction (probability 0.959);
2. the interbank messaging network delivers it (probability 0.995);
3. the central bank's real-time gross settlement system settles it (probability 0.969);
4. the beneficiary bank credits the employee's account (probability 0.993).

The stages are run by different institutions on separate infrastructure, and they succeed or fail independently of one another.

Give every probability to at least four significant figures.

(a) The probability that the payment completes: ______________

(b) The probability that the payment does not complete: ______________

(c) The probability that the payment gets through stages 1 and 2 and then fails at the settlement stage 3:

____

Let W_i be the event that stage i works. A series system works only if every component works, so the event "the payment completes" is the intersection

$$C = W_1 \cap W_2 \cap W_3 \cap W_4.$$

(a) By the multiplicative law (Theorem 2.5), and because the stages are independent so that every conditional probability equals the unconditional one,

$$P(C) = 0.959 \times 0.995 \times 0.969 \times 0.993 = 0.91815.$$

(b) By Theorem 2.7, $P(\overline{C}) = 1 - 0.91815 = 0.08185$.

Composing $\overline{C}$ directly as $\overline{W_1} \cup \cdots \cup \overline{W_4}$ would need the additive law with every intersection term; the complement is far shorter.

(c) The event is $W_1 \cap W_2 \cap \overline{W_3}$ (what happens at stage 4 is then irrelevant, since the payment never reaches it). Again by independence,

$$P(W_1 \cap W_2 \cap \overline{W_3}) = 0.959 \times 0.995 \times 0.031 = 0.954205 \times 0.031 = 0.02958.$$

Every stage here is at least 95% reliable, yet the chain as a whole is less reliable than its weakest link: in a series system each extra stage can only lower $P(C)$.

Problem 4

seed 20260925

A brokerage's trading screens take prices from a primary market-data feed. If the primary feed fails during a session, the system switches to a first backup feed; if that also fails, it switches to a second backup. The three feeds come from different vendors over physically separate lines, and during a session they fail independently with probabilities 0.035 (primary), 0.140 (first backup) and 0.11 (second backup).

Give every probability to at least four significant figures.

(a) The probability that all three feeds fail during a session: ______________

(b) The probability that the screens do not go dark, that is, that the system of three feeds does not completely fail:

____

(c) The probability that the second backup is called into service: ______________

(d) The probability that the session is served by the first backup (the primary fails and the first backup works):

____

Let F_1, F_2, F_3 be the events that the primary, first backup and second backup fail.

(a) The system fails completely only if all three fail, the intersection $F_1 \cap F_2 \cap F_3$. With independence the multiplicative law gives $P(F_1 \cap F_2 \cap F_3) = 0.035 \times 0.140 \times 0.11 = 0.000539$.

(b) "Does not completely fail" is the complement of the event in (a). By Theorem 2.7, $1 - 0.000539 = 0.999461$.

A direct composition, $\overline{F_1} \cup (F_1 \cap \overline{F_2}) \cup (F_1 \cap F_2 \cap \overline{F_3})$, is a union of three mutually exclusive events and gives the same number, but takes three products and a sum instead of one product.

(c) The second backup is used exactly when the primary and the first backup have both failed: $P(F_1 \cap F_2) = 0.035 \times 0.140 = 0.004900$.

(d) $P(F_1 \cap \overline{F_2}) = 0.035 \times 0.860 = 0.030100$.

This is the parallel system: it works if at least one component works, so its failure event is an intersection and redundancy multiplies small numbers together.

Problem 5

seed 20260925

A microfinance lender's scoring system sends an incoming loan application to manual review with probability 0.055. Applications are scored one after another, and whether one is sent to review is independent of every other.

Watch the applications as they arrive on a given morning.

Give every probability to at least four significant figures.

(a) What is the probability that the 12th application is the first one sent to manual review? ______________

(b) What is the probability that at least one of the first 11 applications is sent to manual review? ______________

Let A_i be the event that the ith application is **not** sent to review, so with $p = 0.055$ we have $P(A_i) = 1 - p = 0.945$.

(a) The 12th application is the first one reviewed exactly when the first 11 are not reviewed and the 12th is:

$$E_{12} = A_1 \cap A_2 \cap \cdots \cap A_{11} \cap \overline{A_{12}}.$$

By the multiplicative law, and since the events are independent, every conditional probability equals the unconditional one:

$$P(E_{12}) = (1-p)^{11}p = 0.945^{11} \times 0.055 = 0.536723 \times 0.055 = 0.02952.$$

(b) The complement of "at least one of the first 11 is reviewed" is "none of them is", $A_1 \cap \cdots \cap A_{11}$. By Theorem 2.7,

$$P(\text{at least one}) = 1 - (1-p)^{11} = 1 - 0.53672 = 0.46328.$$

The same number is $P(E_1) + P(E_2) + \cdots + P(E_{11})$, since the E_i are mutually exclusive; summing that geometric series gives exactly $1-(1-p)^{11}$. The complement gets there in one line.

Problem 6

seed 20260925

A bank's trading desk reports a daily 97.5% Value-at-Risk figure. The model is correctly calibrated, so on any trading day the desk's loss exceeds the reported VaR (a "breach") with probability 0.025, and, as the standard backtest assumes, breaches on different days are independent.

Consider the next 24 trading days.

Give every probability to at least four significant figures. You may enter an arithmetic expression such as 0.97^12 instead of a decimal.

(a) The probability of no breach in the 24 days: ________________

(b) The probability of at least one breach in the 24 days: ________________

(c) The probability that each of the first three days is a breach: ________________

Let B_i be the event of a breach on day i, with $P(B_i) = 0.025$.

(a) "No breach" is $\overline{B_1} \cap \overline{B_2} \cap \cdots \cap \overline{B_{24}}$. The complements of independent events are independent, so the multiplicative law gives a product of 24 equal factors:

$$P(\text{no breach}) = (1-0.025)^{24} = 0.975^{24} = 0.54464.$$

(b) "At least one breach" is the union $B_1 \cup \cdots \cup B_{24}$. Expanding that with the additive law would need every intersection of every size; instead, as in Example 2.20, use its complement. By Theorem 2.7

$$P(\text{at least one breach}) = 1 - 0.54464 = 0.45536.$$

(c) $P(B_1 \cap \cdots \cap B_3) = 0.025^3 = 1.563e-05$.

Part (b) is why a correctly calibrated model still produces breaches: over a long enough window "at least one" becomes likely, even though every single day is a small probability. What a backtest tests is whether the number of breaches is too large, not whether breaches occur.

Problem 7

seed 20260925

A bank's internal audit team selects two of the 18 corporate loan files in a branch's portfolio for a detailed review. The first file is drawn at random from all 18; the second is drawn at random from the 17 that remain. The six files with the largest exposures are of particular interest to the auditors; call the other 12 files "small".

Give every probability to at least four significant figures.

(a) One particular large-exposure file is known as file L. Find the probability that file L is drawn first and a small file is drawn second. ________________

(b) Find the probability that exactly one of the six large-exposure files is selected. ________________

(c) Find the probability that both selected files are large-exposure files. ________________

(d) Find the probability that neither selected file is a large-exposure file. ________________

(e) The same week, a different audit team, working independently, draws two of the 22 loan files at a second branch in the same way; that branch also has six large-exposure files. Find the probability that neither audit selects a large-exposure file. ________________

We follow Example 2.19, where two of five applicants are chosen.

(a) Let B_1 be "file L on the first draw" and B_2 "a small file on the second draw". By the multiplicative law (Theorem 2.5),

$$P(B_1 \cap B_2) = P(B_1)P(B_2 \mid B_1) = \frac{1}{18} \cdot \frac{12}{17} = 0.03922.$$

The second factor is conditional: once L has been removed, 12 of the 17 remaining files are small.

(b) "Exactly one large file is selected" is the union of 12 mutually exclusive events: for each of the 6 large files, either it comes first and a small file second, or a small file comes first and it comes second. The first kind has probability 0.03922 as in (a); the second kind has probability $\frac{12}{18} \cdot \frac{1}{17}$, the same number. By the additive law,

$$P(\text{exactly one}) = 12 \times \frac{12}{18 \cdot 17} = \frac{144}{306} = 0.4706.$$

(c) Both large: the first draw is large with probability $6/18$, and given that, the second is large with probability $5/17$:

$$P(\text{both}) = \frac{6}{18} \cdot \frac{5}{17} = \frac{30}{306} = 0.09804.$$

(d) Neither: $\frac{12}{18} \cdot \frac{11}{17} = \frac{132}{306} = 0.4314$.

Check. The three events in (b), (c), (d) are mutually exclusive and exhaust the sample space, so their probabilities must add to 1: $144 + 30 + 132 = 306$. As the book advises, test the composition: here the test is that the pieces add up.

(e) By the reasoning of (d) at the second branch, its audit avoids the large files with probability $\frac{16}{22} \cdot \frac{15}{21} = \frac{240}{462} = 0.5195$. The two audits are carried out independently, so the event "neither audit selects a large file" is the intersection of two independent events:

$$0.4314 \times 0.5195 = 0.2241.$$

Problem 8

seed 20260925

A compliance officer receives a batch of 14 supplier invoices, exactly two of which are forged. She checks the invoices one at a time, in a random order, and a check always reveals whether the invoice in hand is forged.

Give every probability to at least four significant figures.

(a) What is the probability that the second (last) forged invoice is found on the 7th check? ________________

(b) What is the probability that both forged invoices are among the first 4 invoices checked? ________________

(c) Given that exactly one of the two forged invoices was found in the first two checks, what is the probability that the remaining forged invoice is found on the 3rd or the 4th check? ________________

Write F_i for "the ith invoice checked is forged" and G_i for its complement.

(a) The last forgery is found on check 7 exactly when check 7 is forged and exactly one of the first 6 checks is forged. That event is a union of 6 mutually exclusive events, one for each position $i < 7$ of the first forgery:

$$L_7 = \bigcup_{i=1}^{6} \left(F_i \cap F_7 \cap \bigcap_{l<7,\, l \neq i} G_l \right).$$

Apply the multiplicative law to any one of these along the order of checking. The two forged factors contribute 2 and then 1 to the numerator, the genuine ones contribute $12, 11, \ldots$, and the denominators run $14, 13, 12, \ldots$. Every genuine factor cancels against a denominator, leaving

$$\frac{2}{14} \cdot \frac{1}{13} = 0.010989$$

for each i, wherever the first forgery sits. By the additive law,

$$P(L_7) = 6 \times \frac{2}{14 \cdot 13} = \frac{12}{182} = 0.06593.$$

(b) "Both forgeries are among the first 4" is the union of the mutually exclusive events $L_2, L_3, \ldots, L_4$. By part (a),

$$\sum_{j=2}^{4} \frac{2(j-1)}{182} = \frac{4 \cdot 3}{182} = \frac{12}{182} = 0.06593.$$

(c) Once the first two checks have found one forgery, 12 invoices remain and exactly one of them is forged, each position being equally likely. The event is $F_3 \cup (G_3 \cap F_4)$, two mutually exclusive events:

$$\frac{1}{12} + \frac{11}{12} \cdot \frac{1}{11} = \frac{2}{12} = 0.16667.$$

Problem 9

seed 20260925

A treasury clerk must release a payment in the bank's corporate-banking portal and has been given a list of 24 one-time authorisation codes, only one of which is valid for this payment. Having no idea which one it is, the clerk picks a code at random and tries it; if it is rejected, he crosses it off and picks at random from the codes that remain, and so on until the valid code is found.

Give every probability to at least four significant figures.

(a) The probability that the valid code is found on the first try: ________________

(b) The probability that the valid code is found on the 3rd try: ________________

(c) The portal locks the payment if three wrong codes are entered before the valid one. What is the probability that the clerk releases the payment? ________________

(d) Suppose instead the clerk forgets which codes he has tried, so each try is a fresh random pick from all 24 codes (independent of earlier tries), and the portal still locks after three wrong codes. What is the probability that he releases the payment? ________________

Let W_i be "the ith try is wrong" and C_i "the ith try is valid".

(a) All 24 codes are equally likely on the first pick: $P(C_1) = 1/24 = 0.04167$.

(b) The valid code is found on try 3 exactly when the first 2 tries are wrong and try 3 is right: $W_1 \cap \cdots \cap W_2 \cap C_3$. By the multiplicative law, each factor conditioned on what came before,

$$P = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-t+1}{n-t+2} \cdot \frac{1}{n-t+1} = \frac{1}{n},$$

with $n = 24$ and $t = 3$ here, so the probability is $1/24 = 0.04167$.

The product telescopes: each numerator cancels the next denominator. So the valid code is equally likely to turn up on any try, which is also clear by symmetry, since a random order of trying is a random permutation of the list.

(c) The payment is released exactly when the valid code comes up before the 4th try, that is on one of tries $1, \ldots, 3$. These are mutually exclusive events, each of probability $1/24$ by (b), so by the additive law

$$P(\text{released}) = 3 \times \frac{1}{24} = 0.12500.$$

(d) Now the tries are independent, each wrong with probability $23/24 = 0.95833$. The payment is locked exactly when the first 3 tries are all wrong, so by Theorem 2.7 and independence

$$P(\text{released}) = 1 - \left(\frac{23}{24}\right)^3 = 1 - 0.88014 = 0.11986.$$

This is smaller than (c): trying with replacement wastes attempts on codes already known to be wrong. Sampling without replacement makes the events dependent, and here the dependence helps.

Problem 10

seed 20260925

A bank's anti-money-laundering screen raises an alert on 83.5% of illicit transfers and, wrongly, on 19.0% of legitimate ones. An investigator is handed two transfers, T_1 and T_2, and it is known for certain that exactly one of them is illicit (but not which). The screen treats each transfer on its own, so its verdicts on the two are independent.

Give every probability to at least four significant figures.

(a) The probability that the screen raises an alert on both transfers: ________

(b) The probability that it alerts on the illicit transfer and not on the legitimate one: ________

(c) The probability that it is completely wrong: it alerts on the legitimate transfer and not on the illicit one.

____

(d) The probability that it alerts on at least one of the two: ________

Nothing in the answers depends on which of T_1, T_2 is the illicit one, so label them by their status: let I be the event "alert on the illicit transfer" and G the event "alert on the legitimate transfer". Then $P(I) = 0.835$, $P(G) = 0.190$, and I and G are independent.

(a) $P(I \cap G) = P(I)P(G) = 0.835 \times 0.190 = 0.158650$.

(b) $P(I \cap \overline{G}) = 0.835 \times 0.810 = 0.676350$. The complement of an independent event is independent, so the product rule still applies.

(c) $P(\overline{I} \cap G) = 0.165 \times 0.190 = 0.031350$.

(d) "At least one alert" is $I \cup G$. Its complement is $\overline{I} \cap \overline{G}$, so by Theorem 2.7

$$P(I \cup G) = 1 - 0.165 \times 0.810 = 1 - 0.133650 = 0.866350.$$

The additive law gives the same: $0.835 + 0.190 - 0.158650 = 0.866350$.

Check. The events in (a), (b), (c) and "no alert at all" are mutually exclusive and exhaustive, and indeed $0.158650 + 0.676350 + 0.031350 + 0.133650 = 1$.

Problem 11

seed 20260925

A bank launches its new mobile-banking app with an advertising campaign on social media and on television. Market research on the target population finds that a randomly chosen adult sees the social-media ad with probability 0.13, sees the TV ad with probability 0.38, and sees both with probability 0.04. Of those who see at least one of the ads, a proportion 0.38 download the app within a month; of those who see neither ad, a proportion 0.06 download it anyway.

A randomly chosen adult is observed for a month. Let S and T be the events that the person sees the social-media ad and the TV ad, and D the event that the person downloads the app.

Give every probability to at least four significant figures.

(a) $P(S \cup T) =$ ________________

(b) The probability that the person sees neither ad: ________________

(c) $P(D) =$ ________________

(d) The probability that the person downloads the app without having seen either ad: ________________

(a) The additive law (Theorem 2.6): $P(S \cup T) = P(S) + P(T) - P(S \cap T) = 0.13 + 0.38 - 0.04 = 0.47$.

Adding 0.13 and 0.38 alone would count the people who see both ads twice.

(b) “Sees neither” is $\overline{S} \cap \overline{T} = \overline{S \cup T}$ (De Morgan), so by Theorem 2.7 its probability is $1 - 0.47 = 0.53$.

(c) Let $A = S \cup T$. As in Example 2.17, split D over A and its complement:

$$D = (D \cap A) \cup (D \cap \overline{A}),$$

two mutually exclusive events. By the multiplicative law,
$P(D \cap A) = P(D \mid A)P(A) = 0.38 \times 0.47 = 0.1786$ and
$P(D \cap \overline{A}) = P(D \mid \overline{A})P(\overline{A}) = 0.06 \times 0.53 = 0.0318$. By the additive law,

$$P(D) = 0.1786 + 0.0318 = 0.2104.$$

(d) That is $P(D \cap \overline{A}) = 0.0318$, already found in (c).

Notice what the composition needed: the conditional download rates were given for “at least one ad” and “no ad”, so $S \cup T$ had to be computed first. Choosing the composition to match the probabilities you actually know is step 3 of the method.

Problem 12

seed 20260925

A bank has lent to two unrelated borrowers: a construction firm in Baku and an agricultural cooperative in the Ganja region. They operate in different sectors and regions and have no common owners, and the bank treats their defaults over the coming year as independent. Let D_1 be the event that the construction firm defaults, with $P(D_1) = 0.14$, and D_2 the event that the cooperative defaults, with $P(D_2) = 0.07$.

The risk committee asks for the probability of the event E: “exactly one of the two loans defaults”.

Consider the following compositions.

(i) $(D_1 \cap \overline{D_2}) \cup (\overline{D_1} \cap D_2)$

(ii) $D_1 \cup D_2$

(iii) $\overline{D_1} \cap \overline{D_2}$

(iv) $\overline{D_1 \cap D_2}$

(v) $D_1 \cap D_2$

(a) Which composition is exactly the event E (the same set of sample points)?

○ (i)
○ (ii)
○ (iii)
○ (iv)
○ (v)

(b) Find $P(E)$ to at least four significant figures. ________________

The sample space has four points, according to whether each loan defaults: $D_1 \cap D_2$, $D_1 \cap \overline{D_2}$, $\overline{D_1} \cap D_2$, $\overline{D_1} \cap \overline{D_2}$. Test each composition by listing which of these points it contains, as the book recommends after step 3:

- (i) contains the two points with exactly one default;
- (ii) contains every point except “no default”, so it is “at least one”;
- (iii) contains only “no default”;
- (iv) contains every point except “both default”, so it is “at most one”;
- (v) contains only “both default”.

The event asked about, “exactly one of the two loans defaults”, is therefore composition (i).

(b) By independence (and the independence of complements), $P(D_1 \cap D_2) = 0.14 \times 0.07 = 0.0098$, $P(D_1 \cap \overline{D_2}) = 0.14 \times 0.93 = 0.1302$, $P(\overline{D_1} \cap D_2) = 0.86 \times 0.07 = 0.0602$ and $P(\overline{D_1} \cap \overline{D_2}) = 0.86 \times 0.93$. Then:

- (i), additive law on two mutually exclusive events: $0.1302 + 0.0602 = 0.1904$;
- (ii), Theorem 2.6: $0.14 + 0.07 - 0.0098 = 0.2002$;
- (iii), multiplicative law: $0.86 \times 0.93 = 0.7998$;
- (iv), Theorem 2.7: $1 - 0.0098 = 0.9902$;
- (v), multiplicative law: 0.0098.

For the event asked about, $P(E) = 0.1904$.

The common error is to write (ii) for “exactly one” or (v)’s complement for “neither”. Both compositions describe real events, just not the one asked about, and every step after the composition would then be computed correctly for the wrong set.

Problem 13

seed 20260925

A card payment at a shop in Baku is authorised through three electronic components A, B and C. The architecture is as follows.

An authorisation request must first pass the acquirer gateway A. After that, it is sent to the issuer's processor B; if B is down it is sent instead to B's hot-standby processor C, and either B or C can authorise it.

The three components are run by different operators in different data centres and work or fail independently, with probabilities of working $P(A) = 0.97$, $P(B) = 0.87$ and $P(C) = 0.96$ during a given transaction. (Here A, B, C also denote the events that the components work.) Let S be the event that the request is authorised.

Consider the following compositions.

(i) $S = A \cap B \cap C$

(ii) $S = A \cup B \cup C$

(iii) $S = A \cap (B \cup C)$

(iv) $S = (A \cap B) \cup C$

(v) $S = A \cup (B \cap C)$

Give every probability to at least four significant figures.

(a) Which composition describes this architecture?

- ○ (i)
- ○ (ii)
- ○ (iii)
- ○ (iv)
- ○ (v)

(b) $P(S) =$ ________________

(c) The probability that the request cannot be authorised: ________________

(a) Read the description as a statement about which combinations of working components are enough. Components that must all work are joined by ∩ (series); alternatives, any one of which is enough, are joined by ∪ (parallel). Here the correct composition is (iii).

The request succeeds exactly when A works and at least one of B, C works, so S is the intersection of A with the union of B and C. By independence P(S) = P(A) x (1 - P(not B) x P(not C)).

Test it as the book recommends: check a sample point or two. For instance, in this architecture ask whether “only A works” or “only C works” leads to an authorisation, and confirm the chosen composition agrees.

(b) The building blocks, by independence and Theorem 2.7, are $P(B \cup C) = 1 - 0.13 \times 0.04 = 0.9948$, $P(A \cap B) = 0.97 \times 0.87 = 0.8439$ and $P(B \cap C) = 0.87 \times 0.96 = 0.8352$. Then

- (i): $P(A)P(B)P(C) = 0.8439 \times 0.96 = 0.810144$;
- (iii): $P(A)P(B \cup C) = 0.97 \times 0.9948 = 0.964956$;
- (iv): $P(A \cap B) + P(C) - P(A \cap B)P(C) = 0.8439 + 0.96 - 0.8439 \times 0.96 = 0.993756$;
- (v): $P(A) + P(B \cap C) - P(A)P(B \cap C) = 0.97 + 0.8352 - 0.97 \times 0.8352 = 0.995056$.

For the architecture described, composition (iii), $P(S) = 0.964956$.

(c) By Theorem 2.7, $P(\overline{S}) = 1 - 0.964956 = 0.035044$.

Composition (ii) would say that any single component on its own is enough to authorise, which is false in every one of these architectures. Writing the wrong composition in step 3 of the event-composition method is exactly the error the book warns about, and nothing in step 4 will catch it.

Problem 14

seed 20260925

At 90 minutes before the central bank's settlement cut-off, a bank's treasury desk discovers it is short of funds and must borrow overnight from another bank. There are many potential counterparties, and each one, independently of the others, has spare liquidity to lend with probability 0.46. The desk has one dealer, who can check only one counterparty at a time: a check takes 25 minutes and reveals whether that counterparty can lend. As soon as a lender is found, executing and settling the transfer takes a further 20 minutes, and the transfer must be complete by the cut-off.

(a) What is the largest number of counterparties the dealer can check so that, if the last of them turns out to be a lender, the transfer is still complete by the cut-off? ________________

(b) What is the probability that the bank borrows the funds in time? ________________

(c) What is the probability that the bank borrows in time **and** the lender is the last counterparty that could usefully be checked? ________________

Give probabilities to at least four significant figures.

(a) If the jth check is the first to find a lender, the transfer completes $25j + 20$ minutes from now, and this must not exceed 90. So $j \leq (90 - 20)/25$, and the largest whole j is $k = 2$: with $j = 2$ the transfer completes at minute 70, while $j = 3$ would take until minute 95.

(b) Let L_j be the event that the jth counterparty checked can lend. The bank succeeds exactly when at least one of the first 2 can lend:

$$\text{success} = L_1 \cup L_2 \cup \cdots \cup L_2.$$

Its complement is $\overline{L_1} \cap \cdots \cap \overline{L_2}$, whose probability by independence is $0.54^2 = 0.29160$. By Theorem 2.7

$$P(\text{success}) = 1 - 0.29160 = 0.70840.$$

Equivalently, success is the union of the mutually exclusive events “the first lender is the jth checked”, $j = 1, \ldots, 2$, with probabilities $(1-p)^{j-1}p$ (here $p = 0.46$) as in Example 2.21; summing that finite geometric series gives the same answer.

(c) That is the event $\overline{L_1} \cap \cdots \cap \overline{L_1} \cap L_2$, so

$$P = (1-0.46)^1 \times 0.46 = 0.54000 \times 0.46 = 0.24840.$$

Problem 15

seed 20260925

A Baku exporter of dried fruit has three shipments leaving this week to different buyers, each by a different route and carrier: one by rail through Georgia, one by Caspian ferry to Kazakhstan, and one by air freight to the Gulf. Each shipment is delayed beyond its contractual date with probability 0.09, 0.24 and 0.13 respectively, and because the routes, carriers and customs posts are unrelated, delays occur independently.

Give every probability to at least four significant figures.

(a) The probability that no shipment is delayed: ______________

(b) The probability that at least one shipment is delayed: ______________

(c) The probability that exactly one shipment is delayed: ______________

(d) The probability that two or more shipments are delayed: ______________

Let D_i be the event that shipment i is delayed, with $P(D_1) = 0.09$, $P(D_2) = 0.24$, $P(D_3) = 0.13$; the events and their complements are independent.

(a) $P(\overline{D_1} \cap \overline{D_2} \cap \overline{D_3}) = 0.91 \times 0.76 \times 0.87 = 0.601692$.

(b) “At least one delayed” is $D_1 \cup D_2 \cup D_3$, the complement of the event in (a). By Theorem 2.7, $1 - 0.601692 = 0.398308$.

(c) “Exactly one delayed” is the union of three mutually exclusive events,

$$(D_1 \cap \overline{D_2} \cap \overline{D_3}) \cup (\overline{D_1} \cap D_2 \cap \overline{D_3}) \cup (\overline{D_1} \cap \overline{D_2} \cap D_3),$$

with probabilities $0.09 \times 0.76 \times 0.87 = 0.059508$, $0.91 \times 0.24 \times 0.87 = 0.190008$ and $0.91 \times 0.76 \times 0.13 = 0.089908$. By the additive law the probability is 0.339424.

Here the three terms differ, because the delay probabilities differ; one cannot simply multiply one term by 3.

(d) “Two or more”, “exactly one” and “none” are mutually exclusive and exhaust the sample space, so

$$P(\text{two or more}) = 1 - 0.601692 - 0.339424 = 0.058884.$$

Composing “two or more” directly would need four intersections (three with exactly two delays and one with all three); going through complements needs only what has already been computed.

Problem 16

seed 20260925

The SME desk of a bank has two credit officers. For a standard working-capital loan from an established client, officer 1 approves every application on first submission; officer 2 sends every first submission back to the client for additional documents, which the desk records as a first-round rejection. The desk's workload system routes each application to officer 2 with probability 0.455 and to officer 1 otherwise, independently from one application to the next.

This month four established clients each submit one such application, on different days.

Give every probability to at least four significant figures.

(a) What is the probability that exactly three of the four applications are rejected in the first round?

____

(b) What is the probability that all four are approved on first submission? ________________

(c) What is the probability that at least one application is rejected in the first round? ________________

An application is rejected in the first round exactly when it is routed to officer 2. Let R_i be the event that application i is rejected, so $P(R_i) = 0.455$, $P(\overline{R_i}) = 0.545$, and the R_i are independent.

(a) Compose the event as a union of simple sequences. Any particular pattern with 3 rejections and 1 approvals, for example $R_1 \cap \cdots \cap R_3 \cap \overline{R_4} \cap \cdots \cap \overline{R_4}$, has probability, by independence,

$$0.455^3 \times 0.545^1 = 0.051337.$$

Different patterns are mutually exclusive, and there are $\binom{4}{3} = 4$ of them (choose which 3 of the 4 applications are rejected). By the additive law,

$$P(\text{exactly 3 rejected}) = 4 \times 0.051337 = 0.20535.$$

(b) $P(\overline{R_1} \cap \cdots \cap \overline{R_4}) = 0.545^4 = 0.08822$.

(c) This is the complement of (b), so by Theorem 2.7 it is $1 - 0.08822 = 0.91178$.

The applications are identical and the clients are all eligible; which outcome they see depends only on the routing. That is the point of the original Exercise 2.117: here the probability of a rejection measures the process, not the applicants.

Problem 17

seed 20260925

A trader buys a stock and immediately places two orders: a take-profit order above the current price and a stop-loss order below it. The position is closed on the first day on which the price reaches either level. In a simple model of the stock, on each trading day the price reaches the take-profit level with probability 0.110, reaches the stop-loss level with probability 0.035, and otherwise reaches neither; it never reaches both on the same day, and what happens on different days is independent.

Give every probability to at least four significant figures.

(a) What is the probability that the position is closed at the take-profit level on day 4 (the 4th trading day)?

____

(b) What is the probability that the position is still open at the end of day 4? ________________

(c) What is the probability that the take-profit level is reached before the stop-loss level? ________________

(d) What is the probability that the position is closed at the take-profit level within the first 4 trading days?

____

On day i let T_i be "take-profit reached", L_i "stop-loss reached" and N_i "neither reached". These three events are mutually exclusive and exhaustive, so $P(N_i) = 1 - 0.035 - 0.110 = 0.855$.

(a) The position is closed at the take-profit on day 4 exactly when neither level is reached on days $1, \ldots, 3$ and the take-profit is reached on day 4:

$$N_1 \cap \cdots \cap N_3 \cap T_4, \qquad P = 0.855^3 \times 0.110 = 0.625026 \times 0.110 = 0.068753,$$

by independence of days, exactly as in Example 2.21.

(b) Still open after day 4 is $N_1 \cap \cdots \cap N_4$, so $P = 0.855^4 = 0.534398$.

(c) Let E_j be the event of (a) with 4 replaced by j: "closed at take-profit on day j". The event "take-profit first" is the infinite union $E_1 \cup E_2 \cup \cdots$ of mutually exclusive events, so by the additive law (Axiom 3 for countably many events)

$$P(\text{take-profit first}) = \sum_{j=1}^{\infty} 0.855^{j-1} \times 0.110 = \frac{0.110}{1 - 0.855} = \frac{0.110}{0.145} = 0.75862,$$

using the geometric series $\sum_{i \geq 0} x^i = 1/(1-x)$ for $|x| < 1$. The answer is $b/(a+b)$, where $a = 0.035$ and $b = 0.110$ are the daily stop-loss and take-profit probabilities: the days on which nothing happens drop out, and all that matters is the ratio of the two daily probabilities.

(d) Now sum only $j = 1, \ldots, 4$:

$$\sum_{j=1}^{4} 0.855^{j-1} \times 0.110 = 0.110 \cdot \frac{1 - 0.855^4}{1 - 0.855} = 0.75862 \times 0.465602 = 0.35322.$$

That is (c) times the probability that the position has been closed by day 4 at all, $1 - 0.534398$: given that it has closed by then, it closed at the take-profit with the same probability 0.75862 whatever the day.

Problem 18

seed 20260925

A Baku importer pays a supplier in Turkey. The payment leaves the importer's bank S, must reach the supplier's bank R, and in between can travel by either of two correspondent routes:

- **Route 1:** correspondent bank C_1, which passes it on through its own clearing bank H_1 (both must work);
- **Route 2:** correspondent bank C_2, which has a direct account at R.

The payment is sent by route 1; if route 1 fails at either C_1 or H_1, it is re-sent by route 2. It arrives if S works, at least one route works, and R works. On a given day the five institutions work independently, with probabilities

$$P(S) = 0.975,\ P(C_1) = 0.97,\ P(H_1) = 0.93,\ P(C_2) = 0.93,\ P(R) = 0.984,$$

where each letter also denotes the event that the institution works.

Give every probability to at least four significant figures.

(a) The probability that route 1 works: ____________

(b) The probability that at least one of the two routes works: ____________

(c) The probability that the payment arrives: ____________

(d) The probability that the payment does not arrive: ____________

(e) The probability that the payment arrives **only because** of the backup, that is, it arrives and route 1 has failed:

____

(f) Given that the payment arrived, the probability that it went by route 2: ____________

Write $W_1 = C_1 \cap H_1$ for "route 1 works". The event "the payment arrives" is the composition

$$A = S \cap [(C_1 \cap H_1) \cup C_2] \cap R,$$

a series system whose middle block is a parallel pair, one branch of which is itself a series pair.

(a) By independence, $P(W_1) = 0.97 \times 0.93 = 0.9021$.

(b) The block fails only if both routes fail. W_1 depends only on C_1, H_1 and so is independent of C_2; hence by Theorem 2.7

$$P(W_1 \cup C_2) = 1 - (1 - 0.9021)(1 - 0.93) = 1 - 0.0979 \times 0.07 = 0.993147.$$

(c) The three blocks S, $W_1 \cup C_2$, R involve disjoint sets of independent institutions, so they are independent:

$$P(A) = 0.975 \times 0.993147 \times 0.984 = 0.952825.$$

(d) $P(\overline{A}) = 1 - 0.952825 = 0.047175$.

(e) The event is $S \cap \overline{W_1} \cap C_2 \cap R$:

$$P = 0.975 \times 0.0979 \times 0.93 \times 0.984 = 0.087350.$$

(f) The payment went by route 2 exactly when it arrived and route 1 failed, so the event of (e) is a subset of A. By Definition 2.9,

$$P(\overline{W_1} \mid A) = \frac{P(\overline{W_1} \cap A)}{P(A)} = \frac{0.087350}{0.952825} = 0.09168.$$

Notice where the risk sits. The backup lifts the middle block from 0.9021 to 0.993147, but S and R remain single points of failure: even a perfect middle block could not push $P(A)$ above $0.975 \times 0.984 = 0.959400$.

Problem 19

seed 20260925

An exchange's matching engine takes prices from independent market-data feeds, supplied by different vendors over separate lines. Each feed is working at a given moment with probability 0.825, independently of the others.

Give every probability to at least four significant figures.

(a) With three feeds, the engine uses a voting rule: it publishes a price only if at least two of the three feeds are working (so that they can be cross-checked). What is the probability that it can publish? ________________

(b) With the same three feeds, what would that probability be if the engine needed only one working feed?

____

(c) The regulator requires that, under the one-working-feed rule, the probability that at least one feed is working be at least 0.999999. What is the smallest number of feeds that achieves this? ________________

(d) Return to three feeds under the voting rule of (a). The engine also needs at least one of its two matching servers to be working; each server works with probability 0.93, independently of the other and of the feeds. What is the probability that the whole system can publish a price? ________________

Let F_i be the event that feed i works, $P(F_i) = 0.825$, and $P(\overline{F_i}) = 0.175$.

(a) "At least two of three work" is the union of four mutually exclusive events: exactly one of the three patterns $F_1 \cap F_2 \cap \overline{F_3}$, $F_1 \cap \overline{F_2} \cap F_3$, $\overline{F_1} \cap F_2 \cap F_3$, or all three $F_1 \cap F_2 \cap F_3$. Each of the first three has probability $0.825^2 \times 0.175$, so

$$P = 3 \times 0.825^2 \times 0.175 + 0.825^3 = 0.357328 + 0.561516 = 0.918844.$$

(b) The parallel system fails only when every feed fails:
$P(F_1 \cup F_2 \cup F_3) = 1 - 0.175^3 = 1 - 0.005359 = 0.994641$.

The voting rule buys protection against a feed publishing a wrong price, and pays for it in availability: (a) is smaller than (b).

(c) With m feeds, $P(\textbf{at least one works}) = 1 - 0.175^m$, so we need $0.175^m \le 10^{-6}$, that is $m \ge 6/(-\log_{10} 0.175) = 6/0.7570 = 7.926$. The smallest whole number is $m = 8$. Check: $0.175^8 = 8.796 \times 10^{-7}$, which meets the bound, while $0.175^7 = 5.027 \times 10^{-6}$ does not.

(d) The server block is a parallel pair: $1 - 0.07^2 = 0.9951$. The system needs the feed block (a) **and** the server block, a series composition of two independent blocks:

$$P(\text{publish}) = 0.918844 \times 0.9951 = 0.914341.$$

Problem 20

seed 20260925

An external auditor is testing a company's file of 28 expense claims for the year. Unknown to the auditor, exactly 6 of the claims contain an error. The auditor draws claims at random, one at a time and without replacement (a claim already examined is not put back).

Give every probability to at least four significant figures.

(a) The auditor examines a sample of 5 claims. What is the probability that none of them contains an error?

____

(b) What is the probability that the sample of 5 contains at least one error claim? ________________

(c) Suppose instead the auditor keeps drawing until the first error claim is found. What is the probability that the first error claim is the third one examined? ________________

(d) A junior colleague approximates (b) by treating the 5 draws as if they were made with replacement, so that they are independent. What value does the colleague get? ________________

(e) What is the smallest sample size (drawing without replacement) for which the probability of finding at least one error claim is at least 0.90? ________________

Let G_i be the event that the ith claim drawn is error-free and E_i its complement. There are 22 error-free claims.

(a) "No error in the sample" is $G_1 \cap G_2 \cap \cdots \cap G_5$. The draws are **not** independent, so the multiplicative law is applied with conditional probabilities: given that the first i draws were error-free, $22 - i$ of the remaining $28 - i$ claims are error-free. Hence

$$P(G_1 \cap \cdots \cap G_5) = \frac{22}{28} \cdot \frac{21}{27} \cdots \frac{18}{24} = 0.26795.$$

(b) "At least one error claim" is $E_1 \cup \cdots \cup E_5$, the complement of (a). By Theorem 2.7, $1 - 0.26795 = 0.73205$.

(c) The event is $G_1 \cap G_2 \cap E_3$:

$$P = \frac{22}{28} \cdot \frac{21}{27} \cdot \frac{6}{26} = 0.14103.$$

This is Example 2.21's composition, but each factor is conditioned on the draws before it because the file shrinks as claims are removed.

(d) With replacement each draw is error-free with probability $22/28 = 0.7857$, independently, so $1 - 0.7857^5 = 1 - 0.29945 = 0.70055$.

This is smaller than (b). Without replacement, every error-free claim removed makes the remaining file richer in errors, so the true detection probability is higher than the independent-draws approximation.

(e) Increase the sample size one claim at a time, multiplying in the next conditional factor as in (a), until the probability of seeing no error falls to 0.10 or below. With 8 claims the detection probability is 0.89712, short of 0.90; with 9 claims it is 0.92798. The smallest sample size is therefore 9.

Answer key

Problem	Answers
1	(a) 0.2912, (b) 0.0864, (c) 0.3776, (d) 0.3936
2	(a) 0.0196, (b) 0.7396, (c) 0.2408, (d) 0.831617
3	(a) 0.918152, (b) 0.0818477, (c) 0.0295804
4	(a) 0.000539, (b) 0.999461, (c) 0.0049, (d) 0.0301
5	(a) 0.0295197, (b) 0.463277
6	(a) 0.544642, (b) 0.455358, (c) 1.5625E-05
7	(a) 0.0392157, (b) 0.470588, (c) 0.0980392, (d) 0.431373, (e) 0.22409
8	(a) 0.0659341, (b) 0.0659341, (c) 0.166667
9	(a) 0.0416667, (b) 0.0416667, (c) 0.125, (d) 0.119864
10	(a) 0.15865, (b) 0.67635, (c) 0.03135, (d) 0.86635
11	(a) 0.47, (b) 0.53, (c) 0.2104, (d) 0.0318
12	(a) (i), (b) 0.1904
13	(a) (iii), (b) 0.964956, (c) 0.035044
14	(a) 2, (b) 0.7084, (c) 0.2484
15	(a) 0.601692, (b) 0.398308, (c) 0.339424, (d) 0.058884
16	(a) 0.205348, (b) 0.0882239, (c) 0.911776
17	(a) 0.0687529, (b) 0.534398, (c) 0.758621, (d) 0.353216
18	(a) 0.9021, (b) 0.993147, (c) 0.952825, (d) 0.0471748, (e) 0.0873505, (f) 0.0916753
19	(a) 0.918844, (b) 0.994641, (c) 8, (d) 0.914341
20	(a) 0.267949, (b) 0.732051, (c) 0.141026, (d) 0.700551, (e) 9