Week 4, Problem Set 2 - worked solutions

Wackerly 2.10 - total probability and Bayes' rule

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A mobile operator's subscribers are either on a prepaid plan or on a postpaid contract, and nobody is on both. A subscriber is chosen at random. Let B be the event that the subscriber is prepaid and C the event that the subscriber leaves the operator (churns) next month. The operator's records give

$$P(B) = 0.60, \qquad P(C \mid B) = 0.086, \qquad P(C \mid \bar{B}) = 0.013.$$

Give every probability to at least four significant figures.

(a) $P(B \cap C) =$ ______________

(b) $P(\bar{B} \cap C) =$ ______________

(c) $P(C) =$ ______________

Because every subscriber is on exactly one of the two plans, B and $\bar{B}$ satisfy $B \cup \bar{B} = S$ and $B \cap \bar{B} = \emptyset$: the pair $\{B, \bar{B}\}$ is a partition of S in the sense of Definition 2.11, with $k = 2$ and $P(\bar{B}) = 1 - 0.60 = 0.40$.

(a) The multiplicative law (Theorem 2.5): $P(B \cap C) = P(C \mid B)P(B) = 0.086 \times 0.60 = 0.051600$.

(b) Likewise $P(\bar{B} \cap C) = P(C \mid \bar{B})P(\bar{B}) = 0.013 \times 0.40 = 0.005200$.

(c) $C = (C \cap B) \cup (C \cap \bar{B})$, and the two pieces are mutually exclusive because B and $\bar{B}$ are. So their probabilities add, which is Theorem 2.8 with $k = 2$:

$$P(C) = P(C \mid B)P(B) + P(C \mid \bar{B})P(\bar{B}) = 0.051600 + 0.005200 = 0.056800.$$

The overall churn rate is a weighted average of the two plan-specific rates, with the plan shares as weights, so it must lie between 0.013 and 0.086. Of all churners, a fraction $0.051600/0.056800 = 0.9085$ are prepaid -- a first taste of Bayes' rule, which is the subject of the rest of this set.

Problem 2

seed 20260925

A Baku exporter of processed fruit invoices buyers in exactly one of three markets: the EU (B_1), the CIS (B_2) or Asia (B_3). An invoice is chosen at random from last year's book, and L is the event that it was paid late. Of all invoices, a fraction 0.34 went to the EU and 0.33 to the CIS; the rest went to Asia. The late-payment rates by market were

$$P(L \mid B_1) = 0.045, \qquad P(L \mid B_2) = 0.13, \qquad P(L \mid B_3) = 0.17.$$

Give every probability to at least four significant figures.

(a) $P(B_3) =$ ______________

(b) $P(L) =$ ______________

(c) The probability that a randomly chosen invoice was paid on time: ______________

(a) Every invoice goes to exactly one market, so $\{B_1, B_2, B_3\}$ is a partition of the sample space (Definition 2.11): the three sets are mutually exclusive and their union is S. Their probabilities therefore sum to $P(S) = 1$:

$$P(B_3) = 1 - 0.34 - 0.33 = 0.33.$$

(b) All three $P(B_i)$ are positive, so Theorem 2.8 applies with $k = 3$:

$$P(L) = \sum_{i=1}^{3} P(L \mid B_i)P(B_i) = 0.045(0.34) + 0.13(0.33) + 0.17(0.33) = 0.01530 + 0.0429 + 0.0561 = 0.11430.$$

(c) Theorem 2.7: $P(\bar{L}) = 1 - 0.11430 = 0.88570$.

The law of total probability has done here exactly what the book says it is for: $P(L)$ is hard to state directly, but the late-payment rate **within** each market is the natural thing a credit controller tracks, and weighting those by the market shares recovers the overall figure.

Problem 3

seed 20260925

A bank buys the card readers for its ATMs from two suppliers. A fraction 0.47 of its readers come from supplier 1 and the rest from supplier 2. Let B be the event that a reader is from supplier 1 and D the event that it fails its first-year inspection. From service records,

$$P(D \mid B) = 0.038 \qquad \text{and} \qquad P(D \mid \bar{B}) = 0.035.$$

A reader chosen at random fails its inspection. Give every probability to at least four significant figures.

(a) $P(D) =$ ______________

(b) The probability that the failed reader came from supplier 1: $P(B \mid D) =$ ______________

(c) The probability that it came from supplier 2: $P(\bar{B} \mid D) =$ ______________

$\{B, \bar{B}\}$ is a partition of the readers, with $P(B) = 0.47$ and $P(\bar{B}) = 0.53$.

(a) Theorem 2.8 with $k = 2$:

$$P(D) = P(D \mid B)P(B) + P(D \mid \bar{B})P(\bar{B}) = 0.038(0.47) + 0.035(0.53) = 0.01786 + 0.01855 = 0.03641.$$

(b) Bayes' rule, Theorem 2.9: the numerator is the one term of the sum in (a) that belongs to supplier 1, and the denominator is the whole sum.

$$P(B \mid D) = \frac{P(D \mid B)P(B)}{P(D)} = \frac{0.01786}{0.03641} = 0.4905.$$

(c) Either repeat the calculation with the supplier-2 term, $0.01855/0.03641 = 0.5095$, or note that given D the reader came from one supplier or the other, so $P(\bar{B} \mid D) = 1 - 0.4905$.

Compare each posterior with its prior. Observing a failure moves probability towards the supplier with the higher defect rate; how far it moves depends on the ratio of the two defect rates, $0.038/0.035$, and on the market shares together, never on either alone.

Problem 4

seed 20260925

A household survey in Baku asks whether the central bank should cut its policy rate. Let M be the event that a randomly chosen household has a mortgage and F the event that it favours a cut. The survey finds

$$P(M) = 0.20, \qquad P(F \mid M) = 0.88, \qquad P(F \mid \bar{M}) = 0.37.$$

Give every probability to at least four significant figures.

(a) $P(F) =$ ______________

(b) A household that favours a cut is chosen at random. The probability that it has a mortgage, $P(M \mid F) =$ ______________

(c) A household that does **not** favour a cut is chosen at random. The probability that it has a mortgage, $P(M \mid \bar{F}) =$ ______________

$\{M, \bar{M}\}$ partitions the households, with $P(\bar{M}) = 0.80$.

(a) Theorem 2.8:

$$P(F) = 0.88(0.20) + 0.37(0.80) = 0.1760 + 0.2960 = 0.4720.$$

(b) Theorem 2.9:

$$P(M \mid F) = \frac{P(F \mid M)P(M)}{P(F)} = \frac{0.1760}{0.4720} = 0.3729.$$

(c) Now the observed event is $\bar{F}$, so every conditional probability in Bayes' rule is for $\bar{F}$: $P(\bar{F} \mid M) = 1 - 0.88 = 0.12$, and the denominator is $P(\bar{F}) = 1 - 0.4720 = 0.5280$. Then

$$P(M \mid \bar{F}) = \frac{P(\bar{F} \mid M)P(M)}{P(\bar{F})} = \frac{0.12(0.20)}{0.5280} = \frac{0.0240}{0.5280} = 0.0455.$$

Because mortgage holders favour a cut more often than other households, learning that a household favours a cut raises the probability that it has a mortgage above the prior 0.20, and learning that it does not lowers it: $0.0455 < 0.20 < 0.3729$. The prior always lies between the two posteriors -- it is their average, weighted by $P(F)$ and $P(\bar{F})$.

Problem 5

seed 20260925

A loan officer has 30 consumer-loan applications on her desk this week: 15 from salaried workers and 15 from self-employed applicants. Each salaried application is approved with probability 0.78 and each self-employed application with probability 0.62. After the decisions are made, one file is pulled at random from the 30 for a quality review.

Give every probability to at least four significant figures.

(a) The probability that the file pulled is a rejection: ______________

(b) The file pulled is a rejection. The probability that it is from a self-employed applicant: ______________

(c) If instead the file pulled had been an approval, the probability that it is from a self-employed applicant: ______________

Let B_1 and B_2 be the events that the file pulled belongs to a salaried or a self-employed applicant, and R the event that it is a rejection. Every file is in exactly one group, so $\{B_1, B_2\}$ is a partition, and because the file is pulled at random,

$$P(B_1) = \frac{15}{30} = 0.5000, \qquad P(B_2) = \frac{15}{30} = 0.5000.$$

Also $P(R \mid B_1) = 1 - 0.78 = 0.22$ and $P(R \mid B_2) = 1 - 0.62 = 0.38$.

(a) Theorem 2.8:

$$P(R) = 0.22(0.5000) + 0.38(0.5000) = 0.3000.$$

(b) Theorem 2.9:

$$P(B_2 \mid R) = \frac{P(R \mid B_2)P(B_2)}{P(R)} = \frac{0.38(0.5000)}{0.3000} = 0.6333.$$

It is often quicker to cancel the common factor $1/30$ and think in expected counts: about $15 \times 0.38 = 5.70$ self-employed rejections against $15 \times 0.22 = 3.30$ salaried ones, so $5.70/(5.70 + 3.30) = 0.6333$.

(c) Now condition on $\bar{R}$, with $P(\bar{R}) = 1 - 0.3000 = 0.7000$:

$$P(B_2 \mid \bar{R}) = \frac{0.62(0.5000)}{0.7000} = 0.4429,$$

or, in counts, $9.30/(9.30 + 11.70)$.

The self-employed make up 0.5000 of the pile but a larger share of the rejections and a smaller share of the approvals, because they are rejected more often.

Problem 6

seed 20260925

A card issuer's real-time screen flags suspicious transactions. Let F be the event that a transaction is fraudulent and A the event that the screen flags it. The issuer knows

$$P(F) = 0.0020, \qquad P(A \mid F) = 0.98, \qquad P(A \mid \bar{F}) = 0.035.$$

Give every probability to at least four significant figures. Very small answers may be entered in scientific notation with a capital E, e.g. 2.5E-5 (a lowercase e is read as the constant e).

(a) The probability that a randomly chosen transaction is flagged, $P(A) =$ ______________

(b) A transaction has just been flagged. The probability that it is fraudulent, $P(F \mid A) =$ ______________

(c) A transaction was not flagged. The probability that it is nevertheless fraudulent, $P(F \mid \bar{A}) =$ ______________

$\{F, \bar{F}\}$ is a partition with $P(\bar{F}) = 0.9980$.

(a) Theorem 2.8:

$$P(A) = P(A \mid F)P(F) + P(A \mid \bar{F})P(\bar{F}) = 0.98(0.0020) + 0.035(0.9980) = 0.001960 + 0.034930 = 0.036890.$$

(b) Theorem 2.9:

$$P(F \mid A) = \frac{P(A \mid F)P(F)}{P(A)} = \frac{0.001960}{0.036890} = 0.0531.$$

This is the base-rate effect of Exercise 2.125. The screen catches 0.98 of frauds, yet only 0.0531 of flagged transactions are fraudulent: about 947 of every 1000 flags are false alarms. Look at the two terms of the denominator. The false-alarm term 0.034930 is small **per legitimate transaction**, but legitimate transactions outnumber frauds so heavily that it dwarfs the true-alarm term 0.001960.

(c) Condition on $\bar{A}$, with $P(\bar{A} \mid F) = 1 - 0.98 = 0.02$ and $P(\bar{A}) = 1 - 0.036890 = 0.963110$:

$$P(F \mid \bar{A}) = \frac{0.02(0.0020)}{0.963110} = \frac{0.00004000}{0.963110} = 0.00004153.$$

So a transaction that passes the screen is far safer than an average one, while a flagged one is still more likely legitimate than not. The screen is useful as a filter for human review, not as a verdict.

Problem 7

seed 20260925

A bank's corporate loan book is split across four internal rating grades, from grade 1 (safest) to grade 4 (riskiest). A loan is chosen at random; let B_i be the event that it is rated grade i and A the event that it defaults within a year.

	grade 1	grade 2	grade 3	grade 4
$P(B_i)$	0.34	0.25	0.21	0.20
$P(A \mid B_i)$	0.003	0.019	0.049	0.124

Give every probability to at least four significant figures.

(a) The one-year default probability of a randomly chosen loan, $P(A) =$ ______________

(b) A loan has defaulted. The probability that it was rated grade 4, $P(B_4 \mid A) =$ ______________

(c) The probability that a defaulted loan was rated grade 1, $P(B_1 \mid A) =$ ______________

(d) Which grade accounts for the largest share of this year's defaults?

○ Grade 1
○ Grade 2
○ Grade 3
○ Grade 4

Each loan carries exactly one grade, so $\{B_1, B_2, B_3, B_4\}$ is a partition of the loan book (Definition 2.11) and every $P(B_i) > 0$.

(a) Theorem 2.8 with $k = 4$. Compute each term $P(A \mid B_i)P(B_i)$ first, because Bayes' rule in (b)-(d) reuses them:

	grade 1	grade 2	grade 3	grade 4
$P(A \mid B_i)P(B_i)$	0.00102	0.00475	0.01029	0.02480

$$P(A) = 0.00102 + 0.00475 + 0.01029 + 0.02480 = 0.04086.$$

(b) Theorem 2.9: each posterior is its own term divided by the sum,

$$P(B_4 \mid A) = \frac{0.02480}{0.04086} = 0.6070.$$

(c) $P(B_1 \mid A) = 0.00102/0.04086 = 0.0250$.

(d) The four posteriors are 0.0250, 0.1163, 0.2518 and 0.6070, and they sum to 1 because given A the loan still had exactly one grade. The largest is Grade 4's.

The riskiest grade always has the highest default **rate**, but whether it supplies the most **defaults** depends on how much of the book it holds. That is exactly the weighting in the numerator of Bayes' rule: a grade with a modest default rate and a large share of the book can outweigh a small grade with a high rate.

Problem 8

seed 20260925

A consumer lender's scorecard rejects some applicants. Let G be the event that an applicant would repay (a "good" borrower) and R the event that the scorecard rejects the application. From a back-test on past loans,

$$P(\bar{G}) = 0.040, \qquad P(R \mid \bar{G}) = 0.89, \qquad P(R \mid G) = 0.17.$$

Give every probability to at least four significant figures.

(a) $P(R) =$ ______________

(b) The probability that a rejected applicant is a bad borrower, $P(\bar{G} \mid R) =$ ______________

(c) The probability that an accepted applicant is a bad borrower, $P(\bar{G} \mid \bar{R}) =$ ______________

(d) The probability that a rejected applicant is a good borrower, $P(G \mid R) =$ ______________

$\{G, \bar{G}\}$ partitions the applicants, with $P(G) = 0.960$.

(a) Theorem 2.8:

$$P(R) = 0.89(0.040) + 0.17(0.960) = 0.03560 + 0.16320 = 0.19880.$$

(b) Theorem 2.9:

$$P(\bar{G} \mid R) = \frac{P(R \mid \bar{G})P(\bar{G})}{P(R)} = \frac{0.03560}{0.19880} = 0.1791.$$

(c) Now condition on acceptance, using $P(\bar{R} \mid \bar{G}) = 1 - 0.89 = 0.11$ and $P(\bar{R}) = 1 - 0.19880 = 0.80120$:

$$P(\bar{G} \mid \bar{R}) = \frac{0.11(0.040)}{0.80120} = \frac{0.00440}{0.80120} = 0.0055.$$

(d) Given R, the applicant is either good or bad, so $P(G \mid R) = 1 - 0.1791 = 0.8209$.

The scorecard does its job on the accepted pool: the bad rate there, 0.0055, is well below the population rate 0.040. But the majority of the people it turns away would have repaid. The reason is the base rate: the term 0.16320 -- good applicants wrongly rejected -- is built on the large population of good borrowers, and exceeds the term 0.03560 built on the small population of bad ones. That is a cost the lender does not see in its default statistics, only in the profitable loans it never made.

Problem 9

seed 20260925

A bank's anti-money-laundering (AML) system scans outgoing transfers. On historical evidence, 10 transfers in every 10,000 are part of a laundering scheme. Let L be the event that a randomly chosen transfer is one of them and A the event that the system raises an alert on it. The system alerts on a fraction 0.88 of laundering transfers and on a fraction 0.011 of clean ones.

Give every probability to at least four significant figures. Very small answers may be entered in scientific notation with a capital E, e.g. 2.5E-5 (a lowercase e is read as the constant e).

(a) $P(A) =$ ____________

(b) The probability that an alerted transfer really is laundering, $P(L \mid A) =$ ____________

(c) The probability that a transfer with no alert is laundering, $P(L \mid \bar{A}) =$ ____________

(d) The **lift** of the alert, $P(L \mid A)/P(L) =$ ____________

The prior is $P(L) = 10/10000 = 0.0010$, and $\{L, \bar{L}\}$ is a partition with $P(\bar{L}) = 0.9990$. We are told $P(A \mid L) = 0.88$ and $P(A \mid \bar{L}) = 0.011$.

(a) Theorem 2.8:

$$P(A) = 0.88(0.0010) + 0.011(0.9990) = 0.000880 + 0.010989 = 0.011869.$$

(b) Theorem 2.9:

$$P(L \mid A) = \frac{0.000880}{0.011869} = 0.0741.$$

In counts: out of 10,000 transfers there are about 118.7 alerts, of which only about 8.8 are laundering. A compliance officer working the alert queue spends most of the day on clean transfers, even though the system catches most laundering. The denominator is dominated by the clean-transfer term because clean transfers are so numerous.

(c) With $P(\bar{A} \mid L) = 0.12$ and $P(\bar{A}) = 1 - 0.011869 = 0.988131$,

$$P(L \mid \bar{A}) = \frac{0.12(0.0010)}{0.988131} = \frac{0.0001200}{0.988131} = 0.00012144.$$

(d) $0.0741/0.0010 = 74.14$. An alert multiplies the probability of laundering by about 74.14 -- a large amount of evidence -- and still leaves it below one half, because the starting point was so small. "The alert is strong evidence" and "most alerts are false" are both true at once; Bayes' rule is what reconciles them. Equivalently, the lift equals $P(A \mid L)/P(A) = 0.88/0.011869$.

Problem 10

seed 20260925

A pipeline operator lays steel pipe sections bought from three mills. Let B_i be the event that a randomly chosen section came from mill i and D the event that ultrasonic testing finds a weld defect in it.

	mill 1	mill 2	mill 3
$P(B_i)$	0.47	0.34	0.19
$P(D \mid B_i)$	0.016	0.037	0.054

A section has just failed testing. Give every probability to at least four significant figures.

(a) $P(D) =$ ____________

(b) $P(B_1 \mid D) =$ ______________

(c) $P(B_3 \mid D) =$ ______________

(d) Which mill is the most likely source of the defective section?

- ◯ Mill 1
- ◯ Mill 2
- ◯ Mill 3

$\{B_1, B_2, B_3\}$ is a partition with unequal prior probabilities: mill 1 is a priori the most likely source of any section.

(a) Theorem 2.8, keeping the three terms:

$$P(D) = 0.016(0.47) + 0.037(0.34) + 0.054(0.19) = 0.00752 + 0.01258 + 0.01026 = 0.03036.$$

(b) Theorem 2.9:

$$P(B_1 \mid D) = \frac{0.00752}{0.03036} = 0.2477.$$

(c) $P(B_3 \mid D) = 0.01026/0.03036 = 0.3379$, and for completeness $P(B_2 \mid D) = 0.01258/0.03036 = 0.4144$; the three sum to 1.

(d) The largest posterior is Mill 2's. Here the evidence overturns the prior: mill 1 supplies the most sections, but a defective section most likely came from Mill 2, whose defect rate is high enough to outweigh its smaller share.

Bayes' rule multiplies each prior by the likelihood $P(D \mid B_i)$ and renormalizes. The posterior ranking is the ranking of the products $P(D \mid B_i)P(B_i)$, not of either factor alone.

Problem 11

seed 20260925

A motor insurer's underwriters sort policyholders into three risk classes, though the class of an individual driver is never observed directly. Let B_1, B_2, B_3 be the events that a randomly chosen policyholder is low, medium or high risk, and C the event that the policyholder makes at least one claim during the year.

	low	medium	high
$P(B_i)$	0.53	0.29	0.18
$P(C \mid B_i)$	0.045	0.14	0.30

Give every probability to at least four significant figures.

(a) $P(C) =$ ______________

(b) A policyholder claims this year. The probability that she is high risk, $P(B_3 \mid C) =$ ______________

(c) A policyholder has a claim-free year. The probability that he is high risk, $P(B_3 \mid \bar{C}) =$ ______________

(d) $P(B_1 \mid C) =$ ______________

The three classes partition the policyholders, with $P(B_3) = 1 - 0.53 - 0.29 = 0.18$.

(a) Theorem 2.8:

$$P(C) = 0.045(0.53) + 0.14(0.29) + 0.30(0.18) = 0.02385 + 0.0406 + 0.0540 = 0.11845.$$

(b) Theorem 2.9:

$$P(B_3 \mid C) = \frac{0.0540}{0.11845} = 0.4559.$$

(c) The observed event is now $\bar{C}$. Use $P(\bar{C} \mid B_3) = 1 - 0.30 = 0.70$ and $P(\bar{C}) = 1 - 0.11845 = 0.88155$:

$$P(B_3 \mid \bar{C}) = \frac{0.70(0.18)}{0.88155} = \frac{0.1260}{0.88155} = 0.1429.$$

(d) $P(B_1 \mid C) = 0.02385/0.11845 = 0.2014$.

A claim moves the probability of the high-risk class from 0.18 up to 0.4559; a claim-free year moves it down by less, to 0.1429, because claim-free years are common in every class. (The prior is the average of the two posteriors weighted by $P(C)$ and $P(\bar{C})$, and $P(C) < P(\bar{C})$, so the upward move must be the larger.) This asymmetry is the logic of a bonus-malus scale: a claim is informative and earns a large premium increase, while the discount for a clean year is small and has to accumulate over several years. Note from (d) that even among claimants the low-risk class is well represented, simply because it is large.

Problem 12

seed 20260925

A tax authority uses a risk model to select firms for audit. Let E be the event that a firm under-declares its VAT and A the event that the model selects it. A fraction 0.012 of firms under-declare. Following Exercise 2.126, call $P(A \mid E)$ the **sensitivity** of the model, $P(\bar{A} \mid \bar{E})$ its **specificity**, and $P(E \mid A)$ its **positive predictive value** (PPV). The current model has sensitivity 0.89 and specificity 0.90.

Give every probability to at least four significant figures.

(a) The PPV of the current model: ______________

(b) Upgrade A raises the sensitivity to 0.99 and leaves the specificity at 0.90. Its PPV: ______________

(c) Upgrade B leaves the sensitivity at 0.89 and raises the specificity to 0.994. Its PPV: ______________

(d) Keeping the sensitivity at 0.89, what specificity would make the PPV exactly 0.5? ______________

$\{E, \bar{E}\}$ is a partition, with $P(\bar{E}) = 0.988$, and $P(A \mid \bar{E}) = 1-$ specificity. By Theorems 2.8 and 2.9,

$$\text{PPV} = \frac{P(A \mid E)P(E)}{P(A \mid E)P(E) + P(A \mid \bar{E})P(\bar{E})}.$$

(a) $P(A \mid \bar{E}) = 1 - 0.90 = 0.10$:

$$\text{PPV} = \frac{0.89(0.012)}{0.89(0.012) + 0.10(0.988)} = \frac{0.01068}{0.01068 + 0.09880} = 0.0976.$$

(b) Only the numerator's likelihood changes:

$$\text{PPV} = \frac{0.99(0.012)}{0.99(0.012) + 0.10(0.988)} = \frac{0.01188}{0.01188 + 0.09880} = 0.1073.$$

(c) Now the false-selection rate is $1 - 0.994 = 0.006$:

$$\text{PPV} = \frac{0.01068}{0.01068 + 0.006(0.988)} = \frac{0.01068}{0.01068 + 0.005928} = 0.6431.$$

(d) PPV $= 0.5$ means the two terms of the denominator are equal: $P(A \mid \bar{E})P(\bar{E}) = P(A \mid E)P(E)$, so

$$P(A \mid \bar{E}) = \frac{0.89(0.012)}{0.988} = 0.010810, \qquad \text{specificity} = 1 - 0.010810 = 0.989190.$$

This is the answer to Exercise 2.126(c)-(e). The false-selection term $P(A \mid \bar{E})P(\bar{E})$ is multiplied by the large probability $P(\bar{E})$, so when the event is rare the PPV is governed by specificity. Pushing sensitivity up to 0.99 barely moves it, from 0.0976 to 0.1073; cutting the false-selection rate moves it to 0.6431.

Problem 13

seed 20260925

A payments processor scores each online card payment on two signals: S_1, the payment comes from an unusual location for the cardholder, and S_2, it comes from a device never seen before on the card. Let F be the event that the payment is fraudulent, with $P(F) = 0.003$, and

	$P(\cdot \mid F)$	$P(\cdot \mid \bar{F})$
S_1	0.83	0.035
S_2	0.77	0.055

Assume the two signals are conditionally independent given the state -- that is, given F, and again given $\bar{F}$, the events S_1 and S_2 are independent, so for example $P(S_1 \cap S_2 \mid F) = P(S_1 \mid F)P(S_2 \mid F)$. (They are **not** unconditionally independent: both are more common when F occurs.)

Give every probability to at least four significant figures.

(a) A payment shows S_1. $P(F \mid S_1) =$ ____________

(b) The same payment is then found to show S_2 as well. $P(F \mid S_1 \cap S_2) =$ ____________

(c) If instead it had been found **not** to show S_2, $P(F \mid S_1 \cap \bar{S}_2) =$ ____________

Throughout, $\{F, \bar{F}\}$ is the partition, $P(\bar{F}) = 0.997$.

(a) Theorem 2.9 with the first signal:

$$P(F \mid S_1) = \frac{0.83(0.003)}{0.83(0.003) + 0.035(0.997)} = \frac{0.002490}{0.002490 + 0.034895} = 0.066604.$$

(b) Sequential updating. Conditional independence given the state means that, once we know whether the payment is fraudulent, learning S_1 tells us nothing more about S_2: $P(S_2 \mid F \cap S_1) = P(S_2 \mid F) = 0.77$ and $P(S_2 \mid \bar{F} \cap S_1) = 0.055$. So we may apply Bayes' rule again with the posterior from (a) as the new prior:

$$P(F \mid S_1 \cap S_2) = \frac{0.77(0.066604)}{0.77(0.066604) + 0.055(0.933396)} = \frac{0.051285}{0.051285 + 0.051337} = 0.4997.$$

The same number comes from doing it in one step with the joint likelihoods $P(S_1 \cap S_2 \mid F) = 0.83(0.77)$ and $P(S_1 \cap S_2 \mid \bar{F}) = 0.035(0.055)$:

$$\frac{0.003(0.83)(0.77)}{0.003(0.83)(0.77) + 0.997(0.035)(0.055)} = \frac{0.0019173}{0.0019173 + 0.0019192} = 0.4997.$$

(c) Same updating, with $P(\bar{S}_2 \mid F) = 0.23$ and $P(\bar{S}_2 \mid \bar{F}) = 0.945$:

$$P(F \mid S_1 \cap \bar{S}_2) = \frac{0.23(0.066604)}{0.23(0.066604) + 0.945(0.933396)} = \frac{0.015319}{0.015319 + 0.882059} = 0.0171.$$

One signal alone leaves fraud unlikely, 0.0666; two agreeing signals raise it to 0.4997, because each false-alarm rate is small and the chance of two false alarms together is their product. The absence of the second signal pulls the probability back down to 0.0171. The whole calculation leans on the conditional-independence assumption; if the two signals shared a cause other than fraud (a cardholder travelling with a new phone), the product rule in the likelihoods would overstate the evidence.

Problem 14

seed 20260925

At month-end a bank's general ledger does not reconcile, and the break is known to sit in exactly one of three departments: treasury (B_1), payments (B_2) or trade finance (B_3). From past breaks,

$$P(B_1) = 0.39, \qquad P(B_2) = 0.33, \qquad P(B_3) = 0.28.$$

An internal audit of **treasury** finds the break with probability $1 - \alpha = 0.78$ if it is there (so it misses it with probability $\alpha = 0.22$), and of course cannot find it if it is not there. Let U be the event that the audit of treasury is unsuccessful.

Give every probability to at least four significant figures.

The audit of treasury is unsuccessful. What is the probability that the break is in

(a) treasury, $P(B_1 \mid U) =$ ____________

(b) payments, $P(B_2 \mid U) =$ ______________

(c) trade finance, $P(B_3 \mid U) =$ ______________

(d) A second, separate audit of treasury is also unsuccessful. Assume that, given the break is in treasury, the two audits miss it independently, each with probability α. Now what is the probability that the break is in treasury? ______________

The likelihoods of U are $P(U \mid B_1) = \alpha = 0.22$, $P(U \mid B_2) = P(U \mid B_3) = 1$, since an audit of treasury cannot find a break that is elsewhere. By Theorem 2.8,

$$P(U) = 0.22(0.39) + 1(0.33) + 1(0.28) = 0.0858 + 0.61 = 0.6958.$$

(a) Theorem 2.9: $P(B_1 \mid U) = 0.0858/0.6958 = 0.1233$.

(b) $P(B_2 \mid U) = 0.33/0.6958 = 0.4743$.

(c) $P(B_3 \mid U) = 0.28/0.6958 = 0.4024$.

The three sum to 1. An unsuccessful search of treasury lowers its probability and raises the other two **in proportion to their priors**: the ratio $P(B_2 \mid U)/P(B_3 \mid U) = 0.33/0.28$ is unchanged, because the evidence has the same likelihood, 1, under both.

(d) Given B_1, two misses have probability α^2; given B_2 or B_3 they have probability 1. So

$$P(B_1 \mid U_1 \cap U_2) = \frac{0.22^2(0.39)}{0.22^2(0.39) + 0.33 + 0.28} = \frac{0.0189}{0.6289} = 0.0300.$$

The same number comes from updating twice, using the posterior from (a) as the prior for the second audit. Each fruitless audit makes treasury a worse place to spend the next hour of audit time.

Problem 15

seed 20260925

A credit bureau studies small-business borrowers. Among borrowers who went on to default, a fraction 0.45 had a record of at least one payment more than 30 days late in the previous two years. Among borrowers who did not default, the fraction with such a record was only 0.14. The overall default rate is 0.022.

Let D be the event that a randomly chosen borrower defaults and H the event that the borrower has a late-payment record. Give every probability to at least four significant figures.

(a) $P(H) =$ ______________

(b) $P(D \mid H) =$ ______________

(c) $P(D \mid \bar{H}) =$ ______________

(d) The ratio $P(D \mid H)/P(D \mid \bar{H}) =$ ______________

The data are given “the wrong way round”: the study reports $P(H \mid D) = 0.45$ and $P(H \mid \bar{D}) = 0.14$, because it is natural to start from borrowers whose outcome is known and look back at their records. A lender needs the reverse, $P(D \mid H)$, which is what Bayes’ rule supplies. The partition is $\{D, \bar{D}\}$ with $P(\bar{D}) = 0.978$.

(a) Theorem 2.8:

$$P(H) = 0.45(0.022) + 0.14(0.978) = 0.00990 + 0.13692 = 0.14682.$$

(b) Theorem 2.9:

$$P(D \mid H) = \frac{0.00990}{0.14682} = 0.0674.$$

(c) With $P(\bar{H} \mid D) = 0.55$ and $P(\bar{H}) = 1 - 0.14682 = 0.85318$:

$$P(D \mid \bar{H}) = \frac{0.55(0.022)}{0.85318} = \frac{0.01210}{0.85318} = 0.01418.$$

(d) $0.0674/0.01418 = 4.755$.

A late-payment record makes default 4.755 times as likely -- a strong risk factor. Yet a borrower with such a record still defaults with probability only 0.0674, because default is rare to begin with. Do not confuse $P(H \mid D) = 0.45$ ("most defaulters had late payments") with $P(D \mid H)$ ("most borrowers with late payments default"), which is false here.

Problem 16

seed 20260925

Each quarter an equity analyst names which of the 10 sectors of the Baku stock-exchange index will perform best. In any quarter she has a genuinely informed view with probability 0.37; when she does, her pick is right with probability 0.85. Otherwise she guesses, choosing one of the 10 sectors at random, each equally likely.

Let I be the event that she has an informed view this quarter and R the event that her pick is right. Give every probability to at least four significant figures.

(a) $P(R) =$ ______________

(b) Her pick turns out right. The probability that she was informed, $P(I \mid R) =$ ______________

(c) Her pick turns out wrong. The probability that she was informed, $P(I \mid \bar{R}) =$ ______________

$\{I, \bar{I}\}$ is a partition with $P(\bar{I}) = 0.63$. When she guesses uniformly among 10 sectors exactly one of which is best, $P(R \mid \bar{I}) = 1/10 = 0.1000$.

(a) Theorem 2.8:

$$P(R) = 0.85(0.37) + 0.1000(0.63) = 0.3145 + 0.0630 = 0.3775.$$

(b) Theorem 2.9:

$$P(I \mid R) = \frac{0.3145}{0.3775} = 0.8331.$$

(c) $P(\bar{R} \mid I) = 0.15$, $P(\bar{R}) = 1 - 0.3775 = 0.6225$:

$$P(I \mid \bar{R}) = \frac{0.15(0.37)}{0.6225} = \frac{0.0555}{0.6225} = 0.0892.$$

A correct pick raises the probability that she was informed from 0.37 to 0.8331, and the more sectors there are to choose from, the stronger this evidence becomes, because a lucky guess is harder. A wrong pick lowers it to 0.0892. One good quarter is evidence of skill, but not proof of it.

Problem 17

seed 20260925

An analyst models next quarter's macro state as recession (B_1), stagnation (B_2) or expansion (B_3), with

$$P(B_1) = 0.13, \qquad P(B_2) = 0.39, \qquad P(B_3) = 0.48.$$

Let A be the event that a listed bank's share price falls over the quarter and C the event that a cement producer's share price falls. The analyst's conditional probabilities are

	B_1	B_2	B_3
$P(A \mid B_i)$	0.70	0.53	0.19
$P(C \mid B_i)$	0.77	0.46	0.22

and she assumes that **within each macro state** the two stocks move independently: $P(A \cap C \mid B_i) = P(A \mid B_i)P(C \mid B_i)$ for $i = 1, 2, 3$. The macro state is a shared cause of both prices.

Give every probability to at least four significant figures.

(a) $P(A) =$ ______________

(b) $P(A \cap C) =$ ______________

(c) Both stocks fall. The probability that the quarter was a recession, $P(B_1 \mid A \cap C) =$ ______________

(d) The bank stock falls. The probability that the cement stock also falls, $P(C \mid A) =$ ______________

$\{B_1, B_2, B_3\}$ is a partition of the possible quarters.

(a) Theorem 2.8:

$$P(A) = 0.70(0.13) + 0.53(0.39) + 0.19(0.48) = 0.3889.$$

By the same law $P(C) = 0.3851$, which (d) will need.

(b) Theorem 2.8 applies to any event, including $A \cap C$. Within each state the conditional probability factorizes by assumption:

$$P(A \cap C) = \sum_{i=1}^{3} P(A \mid B_i)P(C \mid B_i)P(B_i) = 0.070070 + 0.095082 + 0.020064 = 0.185216.$$

(c) Theorem 2.9 with the event $A \cap C$:

$$P(B_1 \mid A \cap C) = \frac{0.070070}{0.185216} = 0.3783.$$

(d) Definition 2.9:

$$P(C \mid A) = \frac{P(A \cap C)}{P(A)} = \frac{0.185216}{0.3889} = 0.4763.$$

Compare with $P(C) = 0.3851$: $P(C \mid A) > P(C)$, equivalently $P(A \cap C) = 0.185216 > P(A)P(C) = 0.149765$. So A and C are **not** independent, although they are independent within every macro state. The reason is the shared cause: a falling bank stock is evidence of a bad macro state, and a bad state makes the cement stock more likely to fall too. This is why diversifying across sectors of one economy protects much less in a recession than conditional independence might suggest.

Problem 18

seed 20260925

A microfinance lender classifies each borrower as prime (B_1), near-prime (B_2) or subprime (B_3), but a new borrower's true class is unknown. Among new borrowers,

$$P(B_1) = 0.53, \qquad P(B_2) = 0.29, \qquad P(B_3) = 0.18.$$

Let M_1 and M_2 be the events that the borrower misses a scheduled payment in quarters 1 and 2. In each quarter the probability of a missed payment depends on the class only:

$$P(M_j \mid B_1) = 0.020, \qquad P(M_j \mid B_2) = 0.11, \qquad P(M_j \mid B_3) = 0.24, \qquad j = 1, 2.$$

Assume M_1 and M_2 are conditionally independent given the class: for each i, $P(M_1 \cap M_2 \mid B_i) = P(M_1 \mid B_i)P(M_2 \mid B_i)$, and likewise for the complements.

Give every probability to at least four significant figures.

(a) The borrower misses in quarter 1. $P(B_3 \mid M_1) =$ ______________

(b) The borrower misses in both quarters. $P(B_3 \mid M_1 \cap M_2) =$ ______________

(c) Standing at the end of quarter 1, having seen only M_1: the probability of a miss in quarter 2, $P(M_2 \mid M_1) =$ ______________

(d) The borrower misses in quarter 1 but pays in quarter 2. $P(B_3 \mid M_1 \cap \bar{M}_2) =$ ______________

$\{B_1, B_2, B_3\}$ is a partition with unequal priors.

(a) Theorem 2.8 gives the denominator and Theorem 2.9 the posterior:

$$P(M_1) = 0.020(0.53) + 0.11(0.29) + 0.24(0.18) = 0.01060 + 0.03190 + 0.04320 = 0.08570,$$
$$P(B_3 \mid M_1) = \frac{0.04320}{0.08570} = 0.5041.$$

The full posterior after one miss is $(0.1237, 0.3722, 0.5041)$.

(b) By conditional independence the likelihood of two misses in class i is $P(M \mid B_i)^2$. In one step:

$$P(B_3 \mid M_1 \cap M_2) = \frac{0.24^2(0.18)}{0.020^2(0.53) + 0.11^2(0.29) + 0.24^2(0.18)} = \frac{0.010368}{0.0002120 + 0.003509 + 0.010368} = 0.7359.$$

Equivalently, sequentially: take the posterior from (a) as the new prior and multiply by the quarter-2 likelihoods once more,

$$\frac{0.24(0.5041)}{0.020(0.1237) + 0.11(0.3722) + 0.24(0.5041)} = 0.7359.$$

The two routes agree because the common factor $P(M_1)$ cancels -- updating in stages is the same as updating once on all the evidence, **provided** the signals are conditionally independent given the class.

(c) By Definition 2.9 and part (b)'s denominator,

$$P(M_2 \mid M_1) = \frac{P(M_1 \cap M_2)}{P(M_1)} = \frac{0.014089}{0.08570} = 0.1644.$$

Equivalently, it is the posterior-weighted average of the class miss rates, $0.020(0.1237) + 0.11(0.3722) + 0.24(0.5041)$. It exceeds the unconditional $P(M_2) = P(M_1) = 0.08570$: the missed payments are conditionally independent given the class but **dependent** unconditionally, because a first miss is evidence that the borrower is in a riskier class.

(d) The likelihood in class i is $P(M \mid B_i)(1 - P(M \mid B_i))$:

$$P(B_3 \mid M_1 \cap \bar{M}_2) = \frac{0.24(0.76)(0.18)}{0.020(0.980)(0.53) + 0.11(0.89)(0.29) + 0.24(0.76)(0.18)} = \frac{0.032832}{0.010388 + 0.028391 + 0.032832} = 0.4585.$$

A payment made in quarter 2 pulls the subprime probability back from 0.5041 towards the prior; a second miss pushes it up to 0.7359.

Problem 19

seed 20260925

A retail bank studies which customers move their main account to a competitor within a year. Let M be the event that a randomly chosen customer uses the bank's mobile app, Y the event that the customer's salary is paid into the bank, and W the event that the customer switches banks. The two product events overlap:

$$P(M) = 0.45, \qquad P(Y) = 0.58, \qquad P(M \cap Y) = 0.20.$$

The bank's analysts report the switching rate for four customer groups:

group	$P(W \mid \text{group})$
app and salary	0.055
app, no salary	0.09
salary, no app	0.10
neither	0.23

Give every probability to at least four significant figures.

(a) The probability that a customer has neither product, $P(\bar{M} \cap \bar{Y}) =$ ______________

(b) $P(W) =$ ______________

(c) A customer switches. The probability that she had both products: ______________

(d) A customer switches. The probability that he had neither product: ______________

$\{M, Y\}$ is **not** a partition: the two events overlap, and some customers are in neither. Theorem 2.8 needs mutually exclusive sets whose union is the sample space, so the first step is to build them. The four groups in the table do the job:

$$B_1 = M \cap Y, \quad B_2 = M \cap \bar{Y}, \quad B_3 = \bar{M} \cap Y, \quad B_4 = \bar{M} \cap \bar{Y}.$$

Every customer lies in exactly one of them, so $\{B_1, B_2, B_3, B_4\}$ is a partition (Definition 2.11). Their probabilities follow from the three numbers given, using the additive law:

$$P(B_1) = 0.20, \quad P(B_2) = 0.45 - 0.20 = 0.25, \quad P(B_3) = 0.58 - 0.20 = 0.38,$$

(a) $P(B_4) = 1 - P(M \cup Y) = 1 - (0.45 + 0.58 - 0.20) = 0.17$. As a check, the four probabilities sum to 1.

(b) Theorem 2.8 with $k = 4$:

$$P(W) = 0.055(0.20) + 0.09(0.25) + 0.10(0.38) + 0.23(0.17) = 0.01100 + 0.0225 + 0.0380 + 0.0391 = 0.11060.$$

(c) Theorem 2.9: $P(B_1 \mid W) = 0.01100/0.11060 = 0.0995$.

(d) $P(B_4 \mid W) = 0.0391/0.11060 = 0.3535$.

The common mistake is to treat M and Y as if they were a partition and write something like $P(W \mid M)P(M) + P(W \mid Y)P(Y)$. That counts the customers in $M \cap Y$ twice and omits the customers with neither product altogether; there is no reason the result should equal $P(W)$. Before using the law of total probability, check both conditions of Definition 2.11.

Problem 20

seed 20260925

A bank processes international payment orders at 4 regional back offices, each handling the same number of orders. Normally every office makes an error in a fraction 0.028 of orders, the errors scattered at random. In March a software fault at office 1 raised its error rate to 0.075; the other offices were unaffected. The fault was discovered only later.

An auditor receives a batch of March orders, all from one office, but the batch label is lost, so each of the 4 offices is equally likely to be its source. She checks 5 orders from the batch. Let B be the event that the batch came from office 1 and A the event that **exactly one** of the 5 orders checked contains an error.

As in Example 2.23, the probability that exactly j of n orders contain an error, when each does so independently with probability p, is $\binom{n}{j} p^j (1-p)^{n-j}$. Give every probability to at least four significant figures.

(a) $P(A \mid B) =$ ______________

(b) $P(A) =$ ______________

(c) She finds exactly one error. $P(B \mid A) =$ ______________

(d) Suppose instead she had found **exactly two** errors among the 5. Then the probability that the batch came from office 1 would be

____

The partition is $\{B, \bar{B}\}$ with $P(B) = 1/4 = 0.2500$ and $P(\bar{B}) = 3/4 = 0.7500$ -- the other 3 offices can be lumped together because they share the same error rate.

(a) With $p = 0.075$:

$$P(A \mid B) = 5(0.075)(0.925)^4 = 0.274535.$$

(b) Similarly $P(A \mid \bar{B}) = 5(0.028)(0.972)^4 = 0.124966$, and Theorem 2.8 gives

$$P(A) = 0.274535(0.2500) + 0.124966(0.7500) = 0.068634 + 0.093725 = 0.162359.$$

(c) Theorem 2.9:

$$P(B \mid A) = \frac{0.068634}{0.162359} = 0.4227,$$

so $P(\bar{B} \mid A) = 0.5773$.

(d) Now the likelihoods are $\binom{5}{2}(0.075)^2(0.925)^3 = 0.0445192$ and $\binom{5}{2}(0.028)^2(0.972)^3 = 0.0071997$, and

$$P(B \mid \text{two errors}) = \frac{0.0445192(0.2500)}{0.0445192(0.2500) + 0.0071997(0.7500)} = 0.6733.$$

The binomial coefficient cancels, so the data enter the posterior only through the likelihood ratio $\left(\frac{0.075}{0.028}\right)^2\left(\frac{0.925}{0.972}\right)^3$. Each error found multiplies the odds on office 1 by roughly the ratio of the error rates, which is why a second error is far more incriminating than the first. In Example 2.23 one failed fuse raised the probability of line 1 only from 0.20 to 0.37; here one error moves office 1 from 0.2500 to 0.4227, and two errors to 0.6733.

Answer key

Problem	Answers
1	(a) 0.0516, (b) 0.0052, (c) 0.0568
2	(a) 0.33, (b) 0.1143, (c) 0.8857
3	(a) 0.03641, (b) 0.490525, (c) 0.509475
4	(a) 0.472, (b) 0.372881, (c) 0.0454545
5	(a) 0.3, (b) 0.633333, (c) 0.442857
6	(a) 0.03689, (b) 0.0531309, (c) 4.15321E-05
7	(a) 0.04086, (b) 0.606951, (c) 0.0249633, (d) Grade 4
8	(a) 0.1988, (b) 0.179074, (c) 0.00549176, (d) 0.820926
9	(a) 0.011869, (b) 0.0741427, (c) 0.000121441, (d) 74.1427
10	(a) 0.03036, (b) 0.247694, (c) 0.337945, (d) Mill 2
11	(a) 0.11845, (b) 0.455889, (c) 0.14293, (d) 0.201351
12	(a) 0.0975521, (b) 0.107336, (c) 0.643064, (d) 0.98919
13	(a) 0.0666043, (b) 0.499749, (c) 0.0170708
14	(a) 0.123311, (b) 0.474274, (c) 0.402414, (d) 0.0300155
15	(a) 0.14682, (b) 0.0674295, (c) 0.0141822, (d) 4.7545
16	(a) 0.3775, (b) 0.833113, (c) 0.0891566
17	(a) 0.3889, (b) 0.185216, (c) 0.378315, (d) 0.476256
18	(a) 0.504084, (b) 0.735893, (c) 0.164399, (d) 0.458477
19	(a) 0.17, (b) 0.1106, (c) 0.0994575, (d) 0.353526
20	(a) 0.274535, (b) 0.162359, (c) 0.42273, (d) 0.673326