Week 5, Problem Set 1 - worked solutions

Wackerly 2.11-2.13 - random variables and random sampling

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A bank's credit desk tracks three loans over the coming year: a consumer loan, an SME loan and a mortgage, issued to unrelated borrowers in different sectors. Treat their outcomes as independent. The one-year default probabilities are

$$P(\text{consumer defaults}) = 0.04, \quad P(\text{SME defaults}) = 0.14, \quad P(\text{mortgage defaults}) = 0.07.$$

A sample point records the outcome of each loan in the order (consumer, SME, mortgage), for example DND means the consumer loan and the mortgage default and the SME loan does not. Let Y be the number of loans that default.

Give every probability to at least four significant figures.

(a) $P(Y = 1) =$ ______________

(b) $P(Y = 2) =$ ______________

(c) $P(Y \geq 1) =$ ______________

The sample space has $2^3 = 8$ sample points. By independence, the probability of a sample point is the product of the three loans' individual probabilities; for example $P(DND) = 0.04 \times 0.86 \times 0.07$. By Definition 2.12, Y is a function on these eight points, and $P(Y = y)$ is the sum of the probabilities of the points assigned the value y:

$$\{Y = 0\} = \{NNN\}, \quad \{Y = 1\} = \{DNN, NDN, NND\}, \quad \{Y = 2\} = \{DDN, DND, NDD\}, \quad \{Y = 3\} = \{DDD\}.$$

(a) The three points in $\{Y = 1\}$ have probabilities

$$\begin{aligned} P(DNN) &= 0.04 \times 0.86 \times 0.93 = 0.031992, \\ P(NDN) &= 0.96 \times 0.14 \times 0.93 = 0.124992, \\ P(NND) &= 0.96 \times 0.86 \times 0.07 = 0.057792, \end{aligned}$$

so $P(Y = 1) = 0.214776$.

(b) Likewise

$$P(DDN) = 0.005208, \quad P(DND) = 0.002408, \quad P(NDD) = 0.009408,$$

so $P(Y = 2) = 0.017024$.

(c) The complement of $\{Y \geq 1\}$ is the single point NNN, so

$$P(Y \geq 1) = 1 - P(NNN) = 1 - 0.96 \times 0.86 \times 0.93 = 1 - 0.767808 = 0.232192.$$

Because the three loans have different default probabilities, the points inside $\{Y = 1\}$ are not equally likely, and one cannot simply count them: the numerical event is a set of sample points, and its probability is the sum of their probabilities.

Problem 2

seed 20260925

An analyst models the direction of a stock index over the four trading days Monday to Thursday of one week. Each day the index closes either up (U) or down (D), and days are independent. The central bank makes an announcement before trading opens on Wednesday. On Monday and on Tuesday the probability of an up-day is 0.42; from Wednesday onward (on Wednesday and on Thursday) it is 0.68.

A sample point is a sequence of four letters such as $UDDU$. Let Y be the number of up-days in the week.

Give every probability to at least four significant figures.

(a) $P(Y=2)=$ ______________

(b) $P(Y=3)=$ ______________

(c) $P(Y\geq 3)=$ ______________

There are $2^4=16$ sample points. Their probabilities are products of the daily probabilities, but the points are not equally likely because the two halves of the week have different up-probabilities. So $P(Y=y)$ must be found by listing the sample points in the numerical event $\{Y=y\}$ and adding their probabilities.

(a) The event $\{Y=2\}$ contains the $\binom{4}{2}=6$ points with two Us. Group them by where the up-days fall:

- $UUDD$: $0.42^2\times 0.32^2=0.018063$
- $DDUU$: $0.58^2\times 0.68^2=0.155551$
- one up-day early and one late ($UDUD, UDDU, DUUD, DUDU$): each has probability $0.42\times 0.58\times 0.68\times 0.32=0.053007$

Hence $P(Y=2)=0.018063+0.155551+4\times 0.053007=0.385644$.

(b) The event $\{Y=3\}$ has four points:

- $UUUD, UUDU$ (both early days up): each $0.42^2\times 0.68\times 0.32=0.038385$
- $UDUU, DUUU$ (both late days up): each $0.42\times 0.58\times 0.68^2=0.112641$

so $P(Y=3)=2\times 0.038385+2\times 0.112641=0.302051$.

(c) $\{Y\geq 3\}=\{Y=3\}\cup\{Y=4\}$, two mutually exclusive numerical events. $\{Y=4\}=\{UUUU\}$ has probability $0.42^2\times 0.68^2=0.081567$, so $P(Y\geq 3)=0.302051+0.081567=0.383618$.

Problem 3

seed 20260925

A trader holds two contracts that settle at the end of the month, one on Brent crude and one on gold. Treat their outcomes as independent.

- The oil contract pays off with probability 0.44, giving a net gain of 340 AZN; otherwise the trader loses 250 AZN.
- The gold contract pays off with probability 0.66, giving a net gain of 100 AZN; otherwise the trader loses 100 AZN.

Let Y be the trader's total net gain (in AZN) from the two contracts; a loss counts as a negative gain.

Give every probability to at least four significant figures.

(a) $P(Y=240)=$ ______________

(b) $P(Y>0)=$ ______________

(c) $P(Y > -250) =$ ____________

Write W or L for each contract, oil first. The sample space has four points, and Definition 2.12 assigns each one a value of Y:

- WW: $Y = 340 + 100 = 440$, probability $0.44 \times 0.66 = 0.2904$
- WL: $Y = 340 - 100 = 240$, probability $0.44 \times 0.34 = 0.1496$
- LW: $Y = 100 - 250 = -150$, probability 0.56×0.66
- LL: $Y = -250 - 100 = -350$, probability $0.56 \times 0.34 = 0.1904$

(a) The numerical event $\{Y = 240\}$ is the single point WL, so its probability is 0.1496.

(b) $\{Y > 0\} = \{WW, WL\}$: the gold contract's gain 100 is smaller than the oil contract's possible loss 250, so the position is profitable exactly when the oil contract pays off. Therefore

$$P(Y > 0) = 0.2904 + 0.1496 = 0.4400,$$

which is just $P(\text{oil pays off})$.

(c) Of the four values only -350 is below -250, so $\{Y > -250\} = \{WW, WL, LW\}$, the complement of $\{LL\}$:

$$P(Y > -250) = 1 - 0.56 \times 0.34 = 1 - 0.1904 = 0.8096.$$

A statement about the number Y is always a statement about a set of sample points; finding that set is the whole problem.

Problem 4

seed 20260925

The Central Bank's supervision department audits 5 of a commercial bank's 10 branches this quarter. The branches to be audited are chosen by simple random sampling in the sense of Definition 2.13.

(a) How many different samples of branches are possible? ____________

(b) How many of those samples include the Ganja branch? ____________

(c) What is the probability that the Ganja branch is audited? Give at least four significant figures. ____________

(d) Separately, the department inspects 4 of the 16 credit unions on its register, again by simple random sampling. How many different samples of credit unions are possible? ____________

Enter whole numbers in (a), (b) and (d).

(a) A sample is an unordered set of 5 distinct branches, so there are

$$\binom{10}{5} = 252$$

possible samples. Definition 2.13 says the sampling is random exactly when each of them has probability $1/252$.

(b) A sample containing the Ganja branch is fixed by choosing its other 4 members from the remaining 9 branches: $\binom{9}{4} = 126$.

(c) The samples are equally likely, so by the sample-point method

$$P(\text{Ganja audited}) = \frac{\binom{9}{4}}{\binom{10}{5}} = \frac{126}{252} = 0.5000.$$

The ratio simplifies to $n/N = 5/10$: under simple random sampling every unit of the population has the same chance, n/N, of being included.

(d) The same argument with $N = 16$ and $n = 4$ gives $\binom{16}{4} = 1820$ possible samples.

Problem 5

seed 20260925

A relationship manager has 12 corporate accounts and will phone two of them for a satisfaction survey. Two of the accounts belong to the firms Azmetal and Kurstrans.

Give every probability to at least four significant figures.

(a) The two accounts are chosen without replacement, all pairs being equally likely. What is the probability that the sample consists of Azmetal and Kurstrans? ____________

(b) Instead, one account is drawn at random, put back, and a second is drawn at random from all 12. What is the probability that the two accounts drawn are Azmetal and Kurstrans (in either order)? ____________

(c) Under the scheme in (b), what is the probability that the same account is drawn twice? ____________

(d) A payments company picks three of its 38 merchants with replacement, each draw uniform over all 38. What is the probability that the three draws are three different merchants? ____________

(a) Without replacement a sample is an unordered pair of distinct accounts. There are $\binom{12}{2} = 66$ of them, equally likely, and exactly one is {Azmetal, Kurstrans}:

$$P = \frac{1}{66} = 0.015152.$$

(b) With replacement the sample points are ordered pairs, $12^2 = 144$ of them, all equally likely. Two of them, (Azmetal, Kurstrans) and (Kurstrans, Azmetal), give the required sample:

$$P = \frac{2}{144} = 0.013889.$$

This is smaller than (a), because with replacement some probability goes to samples that repeat an account, which cannot happen without replacement. (With $N = 5$ these are the book's $1/10$ and $2/25$.)

(c) The ordered pairs with both entries equal number 12, so $P = 12/144 = 1/12 = 0.083333$.

(d) There are $38^3 = 54872$ equally likely ordered triples. Those with three different merchants number $38 \times 37 \times 36 = 50616$ (the mn rule, Theorem 2.1, applied along the draws), so

$$P = \frac{50616}{54872} = 0.922438.$$

Problem 6

seed 20260925

Among the retail customers of a Baku bank, a fraction 0.35 hold a credit card, a fraction 0.35 hold a consumer loan, and a fraction 0.11 hold both. One customer is selected at random. Let Y be the number of these two products the customer holds, so Y takes the values 0, 1 and 2.

Give every probability to at least four significant figures.

(a) $P(Y = 0) =$ ____________

(b) $P(Y = 1) =$ ____________

(c) $P(Y = 2 \mid Y \geq 1) =$ ____________

Let K be "holds a card" and L "holds a loan". The customers split into four mutually exclusive groups, and Y gives each group one value:

- $K \cap L$ (both): $Y = 2$, probability 0.11
- $K \cap \bar{L}$ (card only): $Y = 1$, probability $0.35 - 0.11 = 0.24$
- $\bar{K} \cap L$ (loan only): $Y = 1$, probability $0.35 - 0.11 = 0.24$
- $\bar{K} \cap \bar{L}$ (neither): $Y = 0$

(a) By the additive law (Theorem 2.6), $P(K \cup L) = 0.35 + 0.35 - 0.11 = 0.59$, and $\{Y = 0\}$ is its complement, so $P(Y = 0) = 1 - 0.59 = 0.41$.

(b) $\{Y = 1\}$ is the union of the two "only" groups, which are mutually exclusive: $P(Y = 1) = 0.24 + 0.24 = 0.48$.

(c) $\{Y = 2\} \subset \{Y \geq 1\}$, so the intersection is $\{Y = 2\}$ and by Definition 2.9

$$P(Y = 2 \mid Y \geq 1) = \frac{P(Y = 2)}{P(Y \geq 1)} = \frac{0.11}{0.59} = 0.18644.$$

As a check, $P(Y = 0) + P(Y = 1) + P(Y = 2) = 0.41 + 0.48 + 0.11 = 1$: the numerical events $\{Y = y\}$ partition the sample space.

Problem 7

seed 20260925

A batch of 7 mortgage files is waiting for an internal audit. Exactly 3 of the files contain documentation errors, but nobody knows which. The auditor examines the files one at a time in a random order (all orderings equally likely) and stops as soon as the last flawed file has been found. Let Y be the number of the inspection at which that happens.

(a) What is the smallest possible value of Y? ____________

Give the probabilities to at least four significant figures.

(b) $P(Y = 4) =$ ____________

(c) $P(Y \leq 5) =$ ____________

(d) $P(Y = 7) =$ ____________

(e) A second batch of 19 files contains exactly one flawed file, and the auditor has time for only 8 inspections, again in a random order. What is the probability that she finds the flawed file? ____________

Only the positions of the flawed files in the inspection order matter. Because every ordering is equally likely, the set of 3 positions they occupy is equally likely to be any of the

$$\binom{7}{3} = 35$$

subsets of $\{1, \ldots, 7\}$. Take these subsets as the sample points. Y is the largest position in the subset.

(a) The 3 flawed files cannot be found in fewer than 3 inspections, and $Y = 3$ happens when they come first. So the smallest value is 3.

(b) $\{Y = 4\}$ consists of the subsets whose largest element is 4: position 4 is in the subset, and the remaining 2 flawed position(s) lie among $1, \ldots, 3$. There are $\binom{3}{2} = 3$ such subsets, so

$$P(Y = 4) = \frac{3}{35} = 0.085714.$$

(c) $\{Y \le 5\}$ is the event that all 3 flawed positions lie in $\{1, \ldots, 5\}$, which happens for $\binom{5}{3} = 10$ subsets:

$$P(Y \le 5) = \frac{10}{35} = 0.28571.$$

(d) $Y = 7$ means the last file inspected is flawed: the remaining 2 flawed position(s) can be anywhere among the first 6, giving $\binom{6}{2} = 15$ subsets and $P(Y = 7) = 15/35 = 0.42857$. This equals $3/7$, the chance that the final file is one of the 3 flawed ones.

(e) With one flawed file the sample points are its 19 equally likely positions, and it is found if its position is among the first 8: $P = 8/19 = 0.42105$. This is the case $D = 1$ of part (c).

Problem 8

seed 20260925

A financial newspaper ranks 13 pension funds from 1 (best five-year return) to 13 (worst). A client who does not read the newspaper picks 5 of the funds at random, every set of 5 funds being equally likely. Let Y be the rank of the best fund in the client's selection (the smallest rank chosen).

(a) How many equally likely selections are there? Enter a whole number. ____________

Give the probabilities to at least four significant figures.

(b) $P(Y = 2) =$ ____________

(c) $P(Y \ge 7) =$ ____________

(a) A selection is a subset of 5 of the 13 ranks, so there are $\binom{13}{5} = 1287$ of them. The client's choice is a random sample in the sense of Definition 2.13, and each has probability $1/1287$.

(b) $\{Y = 2\}$ contains the selections that include rank 2 and whose other 4 funds are all ranked worse than 2, i.e. drawn from the $13 - 2 = 11$ ranks $2 + 1, \ldots, 13$:

$$P(Y = 2) = \frac{\binom{11}{4}}{\binom{13}{5}} = \frac{330}{1287} = 0.25641.$$

(c) $Y \ge 7$ says that none of the ranks $1, \ldots, 6$ was chosen, so all 5 funds come from the 7 ranks $7, \ldots, 13$:

$$P(Y \ge 7) = \frac{\binom{7}{5}}{\binom{13}{5}} = \frac{21}{1287} = 0.016317.$$

Counting the complement directly is much easier here than adding $P(Y = y)$ over $y \ge 7$, though both give the same number.

Problem 9

seed 20260925

The State Tax Service selects 14 of the 31 firms in the construction sector for a detailed audit this year, by simple random sampling (Definition 2.13). Two of the firms are Qala Tikinti and Xazar Beton.

Give every probability to at least four significant figures.

(a) $P(\text{Qala Tikinti is audited}) =$ ____________

(b) $P(\text{both Qala Tikinti and Xazar Beton are audited}) =$ ____________

(c) $P(\text{at least one of the two firms is audited}) =$ ____________

There are $\binom{31}{14}$ equally likely samples. Let A and B be the events that Qala Tikinti and Xazar Beton, respectively, are in the sample.

(a) Samples containing Qala Tikinti are fixed by choosing the other 13 firms from the remaining 30:

$$P(A) = \frac{\binom{30}{13}}{\binom{31}{14}} = \frac{14}{31} = 0.45161.$$

(Write out the two binomial coefficients and all factorials cancel except n/N.)

(b) Samples containing both are fixed by choosing the other 12 firms from the remaining 29:

$$P(A \cap B) = \frac{\binom{29}{12}}{\binom{31}{14}} = \frac{14 \times 13}{31 \times 30} = 0.1957.$$

(c) By the additive law (Theorem 2.6), with $P(B) = P(A)$ by symmetry,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 2 \times 0.45161 - 0.1957 = 0.70753.$$

Note that $P(A \cap B) < P(A)P(B)$: once one firm is in the sample there is one seat fewer for the other, so inclusion events in sampling without replacement are not independent.

Problem 10

seed 20260925

A pension fund's bond portfolio holds 21 bonds, of which 10 are rated high yield (below investment grade). A risk officer reviews 4 bonds chosen at random. Let Y be the number of high-yield bonds among those reviewed.

Give every probability to at least four significant figures.

Sampling **without** replacement (a simple random sample of 4 different bonds):

(a) $P(Y = 0) =$ ______________

(b) $P(Y = 2) =$ ______________

Sampling **with** replacement (4 independent draws, each uniform over all 21 bonds, a bond possibly being drawn more than once; Y counts draws that give a high-yield bond):

(c) $P(Y = 0) =$ ______________

(d) $P(Y = 2) =$ ______________

Without replacement. The $\binom{21}{4} = 5985$ samples are equally likely (Definition 2.13). The portfolio has $21 - 10 = 11$ investment-grade bonds.

(a) $Y = 0$ means all 4 bonds are investment grade:

$$P(Y = 0) = \frac{\binom{11}{4}}{\binom{21}{4}} = \frac{330}{5985} = 0.055138.$$

(b) $Y = 2$: choose 2 of the 10 high-yield bonds and 2 of the 11 others (the mn rule):

$$P(Y = 2) = \frac{\binom{10}{2}\binom{11}{2}}{\binom{21}{4}} = \frac{45 \times 55}{5985} = 0.41353.$$

With replacement. Each draw is high yield with probability $10/21 = 0.4762$, independently of the others.

(c) $P(Y = 0) = (1 - 0.4762)^4 = 0.5238^4 = 0.075282$.

(d) $\{Y = 2\}$ contains the $\binom{4}{2} = 6$ sequences with high-yield bonds at exactly two of the 4 draws, each with probability $0.4762^2 \times 0.5238^2$:

$$P(Y=2)=6\times 0.4762^2\times 0.5238^2=0.3733.$$

The two designs give different probabilities for the same numerical event. Without replacement, each high-yield bond drawn leaves fewer behind, so extreme outcomes such as $Y=0$ are less likely than with replacement. This is the book's point in Section 2.12: the sampling design must be stated, because it changes the probability of the observed sample.

Problem 11

seed 20260925

A currency desk places three trades on the EUR/USD rate during a day. Each trade wins with probability 0.50, independently of the others (they are placed on different, unrelated signals and closed within minutes). A winning trade earns 50 AZN and a losing trade loses 60 AZN. Let Y be the desk's net gain from the three trades, in AZN.

Give every probability to at least four significant figures.

(a) $P(Y=-70)=$ ____________

(b) $P(Y>0)=$ ____________

(c) $P(Y<0)=$ ____________

The sample space has $2^3=8$ points (a W or an L for each trade). Y depends only on the number of wins, so the eight points fall into four numerical events:

- 3 wins (WWW): $Y=3\times 50=150$, probability $0.50^3=0.125000$
- 2 wins (WWL, WLW, LWW): $Y=2\times 50-60=40$, probability $3\times 0.50^2\times 0.50=0.375000$
- 1 win (WLL, LWL, LLW): $Y=50-2\times 60=-70$, probability $3\times 0.50\times 0.50^2=0.375000$
- 0 wins (LLL): $Y=-3\times 60=-180$, probability $0.50^3=0.125000$

Each point in a group has the same probability because of independence, so the probability of a group is (number of points) × (probability of one).

(a) $P(Y=-70)=0.375000$, the "one win" group.

(b) The two-win and three-win values 40 and 150 are positive and -180 is negative. The one-win value $50-2\times 60=-70$ is below zero, so $\{Y>0\}$ excludes the one-win group. Therefore

$$P(Y>0)=0.125000+0.375000=0.500000.$$

(c) Every value of Y is nonzero, so $\{Y<0\}$ is the complement of $\{Y>0\}$ and $P(Y<0)=1-0.500000=0.500000$.

Whether the "one win" outcome counts as profitable depends on the payoff sizes, not on the probabilities; the same sample space supports different numerical events once the function Y changes.

Problem 12

seed 20260925

A mobile operator sells three monthly tariffs: Basic at 9 AZN, Standard at 16 AZN and Premium at 23 AZN. A new subscriber chooses Basic with probability 0.30, Standard with probability 0.38 and Premium with probability 0.32. Two new subscribers, who do not know each other, sign up at a shop today; treat their choices as independent. Let Y be the total monthly revenue (AZN) from these two subscribers.

Give every probability to at least four significant figures.

(a) $P(Y=32)=$ ____________

(b) $P(Y=25)=$ ____________

(c) $P(Y > 32) =$ ______________

A sample point is an ordered pair (first subscriber's tariff, second subscriber's tariff), with letters B, S, P. There are 9 points and, by independence, the probability of a point is the product of the two tariff probabilities. Y assigns each point the sum of the two prices.

(a) The value 32 arises in three ways: (S, S) gives $2 \times 16 = 32$, and (B, P) and (P, B) give $9 + 23 = 32$, because the three prices are equally spaced. So $\{Y = 32\} = \{(S, S), (B, P), (P, B)\}$ and

$$P(Y = 32) = 0.38^2 + 2 \times 0.30 \times 0.32 = 0.1444 + 2 \times 0.0960 = 0.3364.$$

(b) $25 = 9 + 16$ only for (B, S) and (S, B), so $P(Y = 25) = 2 \times 0.30 \times 0.38 = 0.2280$.

(c) The values above 32 come from (S, P), (P, S) and (P, P):

$$P(Y > 32) = 2 \times 0.38 \times 0.32 + 0.32^2 = 2 \times 0.1216 + 0.1024 = 0.3456.$$

The random variable is many-to-one: the event $\{Y = 32\}$ groups three different sample points, which is exactly the partition of S described before Definition 2.12.

Problem 13

seed 20260925

A microfinance lender's scoring model flags loan applications as "high risk". Of all applicants, a proportion 0.04 would default if granted a loan. The model flags 0.93 of the applicants who would default and 0.11 of those who would not. An application is selected at random.

Give every probability to at least four significant figures.

(a) What is the probability that it is flagged? ______________

(b) Given that it is flagged, what is the probability that the applicant would default? ______________

(c) Given that it is not flagged, what is the probability that the applicant would default? ______________

Let D be "would default" and F "flagged". We are given $P(D) = 0.04$, $P(F \mid D) = 0.93$ and $P(F \mid \bar{D}) = 0.11$. The events D and $\bar{D}$ partition the applicants.

(a) By the law of total probability (Theorem 2.8),

$$P(F) = P(F \mid D)P(D) + P(F \mid \bar{D})P(\bar{D}) = 0.93 \times 0.04 + 0.11 \times 0.96 = 0.0372 + 0.1056 = 0.1428.$$

(b) By Bayes' rule (Theorem 2.9),

$$P(D \mid F) = \frac{P(F \mid D)P(D)}{P(F)} = \frac{0.0372}{0.1428} = 0.2605.$$

(c) $P(\bar{F} \mid D) = 1 - 0.93 = 0.07$, and $P(\bar{F}) = 1 - 0.1428 = 0.8572$, so

$$P(D \mid \bar{F}) = \frac{P(\bar{F} \mid D)P(D)}{P(\bar{F})} = \frac{0.07 \times 0.04}{0.8572} = \frac{0.0028}{0.8572} = 0.0032664.$$

About 26.1 percent of flagged applicants would actually default, which is below the 0.93 of defaulters the model catches: $P(D \mid F)$ and $P(F \mid D)$ are different conditional probabilities. Because defaulters are a small share of applicants, the false flags among the large group of good applicants make up a substantial part of the flagged pool. An unflagged applicant, on the other hand, is very safe.

Problem 14

seed 20260925

A card-payment processor connects its Baku data centre to the interbank switch through four leased links. Links 1 and 2 are supplied by vendor X and each is up on a given day with probability 0.71; links 3 and 4 are supplied by vendor Y, on a separate cable route, and each is up with probability 0.89. The four links fail independently.

Two wiring designs are considered.

- **Design A:** two separate routes, one through links 1 then 2, the other through links 3 then 4. Traffic gets through if at least one route has both its links up.
- **Design B:** a first stage made of links 1 and 3 side by side, followed by a second stage made of links 2 and 4 side by side. Traffic gets through if at least one link in each stage is up.

Give every probability to at least four significant figures.

(a) $P(\text{traffic gets through under design A}) =$ ______________

(b) $P(\text{traffic gets through under design B}) =$ ______________

(c) Under design A, given that traffic gets through, what is the probability that links 1 and 4 are both up? ______________

Let U_i be the event that link i is up.

(a) Route 1-2 works with probability $P(U_1 \cap U_2) = 0.71^2 = 0.5041$ and route 3-4 with probability $0.89^2 = 0.7921$ (multiplicative law with independence). Design A fails only if both routes fail, and the routes use different links, so

$$P(A) = 1 - (1 - 0.5041)(1 - 0.7921) = 0.8969.$$

(b) A stage fails only if both its links are down, so each stage works with probability $1 - 0.29 \times 0.11 = 0.9681$. The two stages use different links, hence are independent:

$$P(B) = 0.9681^2 = 0.9372.$$

Design B is the more reliable one: it can use route 1-4 and route 3-2 as well as the two routes design A has.

(c) By Definition 2.9, $P(U_1 \cap U_4 \mid A) = P(U_1 \cap U_4 \cap A)/P(A)$. If links 1 and 4 are up, design A works exactly when link 2 is up (completing route 1-2) or link 3 is up (completing route 3-4). So

$$P(U_1 \cap U_4 \cap A) = 0.71 \times 0.89 \times [1 - 0.29 \times 0.11] = 0.6319 \times 0.9681 = 0.6117,$$

and

$$P(U_1 \cap U_4 \mid A) = \frac{0.6117}{0.8969} = 0.6821.$$

Conditioning on the system working raises the probability that any given pair of links is up, compared with the unconditional 0.6319.

Problem 15

seed 20260925

A regional bond fund's approved list contains 5 Azerbaijani, 11 Turkish and 5 Kazakh sovereign and quasi-sovereign bonds, all different. For a new sub-portfolio the manager selects 4 bonds from the list by simple random sampling.

Give every probability to at least four significant figures.

(a) What is the probability that all three countries are represented? ______________

(b) What is the probability that all four bonds come from the same country? _______________

(c) Given that all three countries are represented, what is the probability that two of the four bonds are Azerbaijani?

____

The list has $5 + 11 + 5 = 21$ bonds, so there are $\binom{21}{4} = 5985$ equally likely samples (Definition 2.13). Each probability is a count of samples divided by 5985.

(a) With 4 bonds and three countries all present, one country contributes 2 bonds and the other two contribute 1 each. By the mn rule the counts are

- two Azerbaijani: $\binom{5}{2} \times 11 \times 5 = 10 \times 11 \times 5 = 550$
- two Turkish: $5 \times \binom{11}{2} \times 5 = 1375$
- two Kazakh: $5 \times 11 \times \binom{5}{2} = 550$

These cases are mutually exclusive, so

$$P(\text{all three countries}) = \frac{550 + 1375 + 550}{5985} = \frac{2475}{5985} = 0.41353.$$

(b) $\binom{5}{4} + \binom{11}{4} + \binom{5}{4} = 5 + 330 + 5 = 340$ samples are from a single country (a coefficient is 0 when a country has fewer than 4 bonds), so $P = 340/5985 = 0.056809$.

(c) Let T be "all three countries represented" and A_2 "two Azerbaijani bonds". Here $A_2 \cap T$ is the first case of (a), so by Definition 2.9

$$P(A_2 \mid T) = \frac{P(A_2 \cap T)}{P(T)} = \frac{550/5985}{2475/5985} = \frac{550}{2475} = 0.22222.$$

Given T, we simply count within the 2475 samples that make up T.

Problem 16

seed 20260925

A motor insurer receives claims on policies sold through three channels. A proportion 0.43 of claims come from policies sold by agents, 0.33 from policies bought online and 0.24 from policies sold by brokers. The proportion of claims that turn out to be fraudulent is 0.020 for agent-sold policies, 0.075 for online policies and 0.035 for broker-sold policies. One claim is selected at random.

Give every probability to at least four significant figures.

(a) $P(\text{the claim is fraudulent}) =$ _______________

(b) Given that the claim is fraudulent, what is the probability that the policy was bought online? _______________

(c) Given that the policy was not sold by an agent, what is the probability that the claim is fraudulent? _______________

Let A, O, B be the channels (a partition of the claims) and F the event that the claim is fraudulent.

(a) Law of total probability (Theorem 2.8):

$$P(F) = P(F \mid A)P(A) + P(F \mid O)P(O) + P(F \mid B)P(B)$$

$$= 0.020 \times 0.43 + 0.075 \times 0.33 + 0.035 \times 0.24 = 0.00860 + 0.02475 + 0.00840 = 0.04175.$$

(b) Bayes' rule (Theorem 2.9):

$$P(O \mid F) = \frac{P(F \mid O)P(O)}{P(F)} = \frac{0.02475}{0.04175} = 0.5928.$$

(c) $\bar{A} = O \cup B$, with $P(\bar{A}) = 0.33 + 0.24 = 0.57$. Because O and B are mutually exclusive, $P(F \cap \bar{A}) = P(F \cap O) + P(F \cap B) = 0.02475 + 0.00840 = 0.03315$, and by Definition 2.9

$$P(F \mid \bar{A}) = \frac{0.03315}{0.57} = 0.05816.$$

Note that $P(F \mid \bar{A})$ is a weighted average of 0.075 and 0.035, with weights proportional to the channel shares; it is not their simple average.

Problem 17

seed 20260925

An equity analyst ranks the 10 listed bank stocks she covers from 1 (most attractive) to 10 (least). An investor who ignores the report buys 4 of the stocks chosen at random, every set of 4 being equally likely. Let R be the range of the ranks bought, $R =$ (largest rank) $-$ (smallest rank).

Give the probabilities to at least four significant figures.

(a) What is the probability that the best rank bought is 2 and the worst is 7? ______________

(b) What is the smallest possible value of R? ______________

(c) $P(R = 5) =$ ______________

(d) $P(R \leq 5) =$ ______________

The sample points are the $\binom{10}{4} = 210$ equally likely sets of 4 ranks.

(a) The set must contain 2 and 7, and its remaining 2 rank(s) must lie strictly between them, among the $7 - 2 - 1 = 4$ ranks $3, \ldots, 6$:

$$P = \frac{\binom{4}{2}}{210} = \frac{6}{210} = 0.028571.$$

(b) 4 different ranks span at least 4 consecutive integers, so $R \geq 4 - 1 = 3$, with equality for a block of consecutive ranks.

(c) For $R = 5$ the smallest rank m can be any of $1, \ldots, 10 - 5$, i.e. 5 choices, and the largest is then $m + 5$. For each such pair, exactly as in (a), the remaining 2 rank(s) come from the 4 ranks in between. By the mn rule

$$P(R = 5) = \frac{(10 - 5)\binom{4}{2}}{210} = \frac{5 \times 6}{210} = 0.14286.$$

(d) The same argument gives $P(R = s) = (N - s)\binom{s-1}{n-2} / \binom{N}{n}$ for every $s \geq n - 1$. The events $\{R = s\}$ are mutually exclusive, so

$$P(R \leq 5) = \sum_{s=3}^{5} \frac{(10 - s)\binom{s-1}{2}}{210} = \frac{55}{210} = 0.2619.$$

Problem 18

seed 20260925

A bank's Sumqayit branch holds 12 pending mortgage applications, 5 of them prime and 7 subprime. The Baku head office holds 5 prime and 10 subprime applications. One application is chosen at random at Sumqayit and sent to head office. A credit officer at head office then picks one application at random from everything now on her desk.

Give every probability to at least four significant figures.

(a) What is the probability that the officer picks a subprime application? ______________

(b) Given that she picks a subprime application, what is the probability that the application sent from Sumqayit was prime?

____

(c) Given that she picks a prime application, what is the probability that the application sent from Sumqayit was prime?

____

Let T_p and T_s be the events that the transferred application is prime or subprime (a partition), and S, P the events that the officer's pick is subprime or prime. After the transfer head office holds $5+10+1=16$ applications.

- $P(T_p)=5/12$, and then head office has 10 subprime, so $P(S \mid T_p)=10/16$.
- $P(T_s)=7/12$, and then head office has 11 subprime, so $P(S \mid T_s)=11/16$.

(a) Law of total probability (Theorem 2.8):

$$P(S)=\frac{5}{12}\cdot\frac{10}{16}+\frac{7}{12}\cdot\frac{11}{16}=\frac{50+77}{192}=\frac{127}{192}=0.6615.$$

(b) Bayes' rule (Theorem 2.9):

$$P(T_p \mid S)=\frac{P(S \mid T_p)P(T_p)}{P(S)}=\frac{50/192}{127/192}=\frac{50}{127}=0.3937.$$

(c) Now $P(P \mid T_p)=6/16$ and $P(P \mid T_s)=5/16$, so

$$P(T_p \mid P)=\frac{5\times 6}{5\times 6+7\times 5}=\frac{30}{65}=0.4615.$$

Compare both with the prior $P(T_p)=0.4167$. Seeing a subprime pick lowers the probability that a prime file was sent, and seeing a prime pick raises it: the transferred file makes the pick slightly more likely to share its type, so the observation is evidence about the first stage, even though it happened later.

Problem 19

seed 20260925

A proprietary trader's record over 11 trading days shows 7 gain days (G) and 4 loss days (L). If the trader's results carry no momentum, every arrangement of the 4 loss days among the 11 days is equally likely. A **run** is a maximal unbroken block of equal letters; for example $GGLLGG\ldots$ starts with a run of Gs. Let R be the number of runs in the record.

Give every probability to at least four significant figures.

(a) What is the probability of the arrangement in which the 4 loss days are the last 4 days of the record? ______________

(b) $P(R=2)=$ ______________

(c) $P(R\le 3)=$ ______________

(d) A second trader's record covers 13 days with 5 loss days, again with all arrangements equally likely. What is the probability that the 5 loss days are consecutive (form a single block)? ______________

An arrangement is fixed by which 4 of the 11 days are loss days, so there are $\binom{11}{4}=330$ equally likely sample points. R is a function on them.

(a) The arrangement $G\cdots G\,L\cdots L$ is one sample point: $P=1/330=0.0030303$.

(b) With both letters present, $R\ge 2$, and $R=2$ means one block of each letter: $G\cdots G\,L\cdots L$ or $L\cdots L\,G\cdots G$. So $P(R=2)=2/330=0.0060606$.

(c) $R=3$ happens in two ways.

- One block of Ls sits inside the Gs, splitting the 7 gain days into two nonempty blocks. The split can be made in $7-1=6$ places.
- One block of Gs sits inside the Ls, splitting the 4 loss days into two nonempty blocks: $4-1=3$ ways.

So 9 points have $R=3$, and

$$P(R \le 3) = \frac{2+9}{330} = \frac{11}{330} = 0.033333.$$

(d) There are $\binom{13}{5} = 1287$ arrangements. A single block of 5 consecutive days can start on any of days $1, \ldots, 13-5+1$, so 9 arrangements qualify:

$$P = \frac{9}{1287} = 0.006993.$$

Small probabilities like these are how the book's Exercise 2.160 turns "the losses came in a cluster" into evidence against randomness.

Problem 20

seed 20260925

A commodities desk models Brent crude over the next two trading days. On day 1 the price rises with probability 0.37 and otherwise falls; on day 2 it rises with probability 0.66 and otherwise falls. The two days' moves are independent. Define the events

A: the price rises on day 1.

B: the price rises on day 2.

C: the price moves in the same direction on both days.

Give the probabilities to at least four significant figures.

(a) $P(C) =$ ______________

(b) $P(A \cap C) =$ ______________

(c) Are A and C independent?

○ Independent
○ Dependent

(d) Are B and C independent?

○ Independent
○ Dependent

(e) Are A, B and C mutually independent (all three pairwise product rules and $P(A \cap B \cap C) = P(A)P(B)P(C)$)?

○ Yes
○ No

Write U (up) or D (down) for each day. The four sample points have probabilities $P(UU) = 0.37 \times 0.66$, $P(UD) = 0.37 \times 0.34$, $P(DU) = 0.63 \times 0.66$, $P(DD) = 0.63 \times 0.34$. As sets, $A = \{UU, UD\}$, $B = \{UU, DU\}$, $C = \{UU, DD\}$.

(a) $P(C) = P(UU) + P(DD) = 0.37 \times 0.66 + 0.63 \times 0.34 = 0.4584$.

(b) $A \cap C = \{UU\}$, so $P(A \cap C) = 0.37 \times 0.66 = 0.2442$.

(c) Definition 2.10 needs $P(A \cap C) = P(A)P(C) = 0.37 \times 0.4584 = 0.1696$. This is not equal to 0.2442, so A and C are dependent. In general

$$P(A)P(C) - P(A \cap C) = p_1(p_1 p_2 + q_1 q_2) - p_1 p_2 = p_1 q_1 (q_2 - p_2),$$

which is zero exactly when $p_2 = 1/2$: whether "same direction" tells you anything about day 1 depends only on whether day 2 is a fair coin.

(d) By the same algebra with the days exchanged, $P(B)P(C) = 0.66 \times 0.4584 = 0.3025$ against $P(B \cap C) = 0.2442$, so B and C are dependent (p_1 must be $1/2$).

(e) $A \cap B \cap C = \{UU\}$, so $P(A \cap B \cap C) = 0.2442$, while $P(A)P(B)P(C) = 0.37 \times 0.66 \times 0.4584 = 0.1119$. These can only agree if $P(C) = 1$, which is impossible here, so the three events are never mutually independent. Even with two fair days, when every pair is independent, knowing A and B together determines C: pairwise independence does not imply mutual independence.

Answer key

Problem	Answers
1	(a) 0.214776, (b) 0.017024, (c) 0.232192
2	(a) 0.385644, (b) 0.302051, (c) 0.383618
3	(a) 0.1496, (b) 0.44, (c) 0.8096
4	(a) 252, (b) 126, (c) 0.5, (d) 1820
5	(a) 0.0151515, (b) 0.0138889, (c) 0.0833333, (d) 0.922438
6	(a) 0.41, (b) 0.48, (c) 0.186441
7	(a) 3, (b) 0.0857143, (c) 0.285714, (d) 0.428571, (e) 0.421053
8	(a) 1287, (b) 0.25641, (c) 0.016317
9	(a) 0.451613, (b) 0.195699, (c) 0.707527
10	(a) 0.0551378, (b) 0.413534, (c) 0.0752824, (d) 0.373301
11	(a) 0.375, (b) 0.5, (c) 0.5
12	(a) 0.3364, (b) 0.228, (c) 0.3456
13	(a) 0.1428, (b) 0.260504, (c) 0.00326645
14	(a) 0.896902, (b) 0.937218, (c) 0.682061
15	(a) 0.413534, (b) 0.0568087, (c) 0.222222
16	(a) 0.04175, (b) 0.592814, (c) 0.0581579
17	(a) 0.0285714, (b) 3, (c) 0.142857, (d) 0.261905
18	(a) 0.661458, (b) 0.393701, (c) 0.461538
19	(a) 0.0030303, (b) 0.00606061, (c) 0.0333333, (d) 0.00699301
20	(a) 0.4584, (b) 0.2442, (c) Dependent, (d) Dependent, (e) No