Week 8, Problem Set 1 - worked solutions

Wackerly 3.9 - Moments and Moment-Generating Functions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A retail brokerage in Baku records Y, the number of trades a randomly chosen active client places on a given day. From last quarter's data,

y	0	1	2	3
$p(y)$	0.47	0.24	0.24	0.05

(a) Using Definition 3.14, write the moment-generating function of Y as a function of t. (Type e^{2t} as e^(2*t), for example.)

$m(t) =$ ______________

(b) Use Theorem 3.12 to find $E(Y) = m^{(1)}(0) =$ ______________

(c) Use Theorem 3.12 to find $E(Y^2) = m^{(2)}(0) =$ ______________

(a) Definition 3.14 says $m(t) = E(e^{tY})$. For a discrete Y this expectation is the sum of $e^{ty}p(y)$ over the support (Theorem 3.2 with $g(Y) = e^{tY}$):

$$m(t) = \sum_y e^{ty}p(y) = 0.47e^0 + 0.24e^t + 0.24e^{2t} + 0.05e^{3t} = 0.47 + 0.24e^t + 0.24e^{2t} + 0.05e^{3t}.$$

Each value y of the random variable appears as the exponent e^{yt}, and its probability as the coefficient. Check: $m(0) = 0.47 + 0.24 + 0.24 + 0.05 = 1$, as it must for every moment-generating function.

(b) Differentiate term by term:

$$m^{(1)}(t) = 0.24e^t + 0.48e^{2t} + 0.15e^{3t}.$$

At $t = 0$ every exponential equals 1, so by Theorem 3.12

$$E(Y) = \mu_1' = m^{(1)}(0) = 0.24 + 0.48 + 0.15 = 0.87.$$

(c) Differentiate once more:

$$m^{(2)}(t) = 0.24e^t + 0.96e^{2t} + 0.45e^{3t},$$

$$E(Y^2) = \mu_2' = m^{(2)}(0) = 0.24 + 0.96 + 0.45 = 1.65.$$

These agree with the direct sums $\sum y\,p(y)$ and $\sum y^2 p(y)$, as they must: each derivative brings down one more factor of y, and setting $t = 0$ removes the exponentials. With Theorem 3.6 the daily trade count has variance $\sigma^2 = 1.65 - 0.87^2 = 0.8931$.

Problem 2

seed 20260925

A dealer quotes a benchmark Azerbaijani government bond. Let Y be the change in its price from one day's close to the next, measured in ticks. The dealer's model is

y	-1	0	1	2
$p(y)$	0.13	0.46	0.33	0.08

Use the book's notation: $\mu'_k = E(Y^k)$ is the kth moment about the origin (Definition 3.12) and $\mu_k = E[(Y-\mu)^k]$ the kth central moment (Definition 3.13). Give answers to at least four significant figures.

(a) $\mu'_1 =$ ______________

(b) $\mu'_2 =$ ______________

(c) $\mu'_3 =$ ______________

(d) $\mu_2 =$ ______________

(e) $\mu_3 =$ ______________

Moments about the origin are expectations of powers of Y, so each one is a sum $\sum_y y^k p(y)$ (Theorem 3.2).

(a) $\mu'_1 = (-1)(0.13) + 0(0.46) + 1(0.33) + 2(0.08) = -0.13 + 0.33 + 0.16 = 0.36$.

(b) $\mu'_2 = (-1)^2(0.13) + 1^2(0.33) + 2^2(0.08) = 0.13 + 0.33 + 0.32 = 0.78$.

(c) $\mu'_3 = (-1)^3(0.13) + 1^3(0.33) + 2^3(0.08) = -0.13 + 0.33 + 0.64 = 0.84$.

Odd powers keep the sign of y, so the down-tick day enters (a) and (c) with a minus sign; even powers do not, so it enters (b) with a plus sign.

(d) The second central moment is the variance: $\mu_2 = \sigma^2$. By Theorem 3.6,

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = 0.78 - (0.36)^2 = 0.6504.$$

(e) Expand $(Y-\mu)^3 = Y^3 - 3\mu Y^2 + 3\mu^2 Y - \mu^3$ and take expectations term by term (Theorem 3.5):

$$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 3\mu^2\mu - \mu^3 = \mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3$$

$$\mu_3 = 0.84 - (0.8424) + (0.0933) = 0.0909.$$

(You get the same number from the definition directly, $\sum_y (y - 0.36)^3 p(y)$.) The third central moment is positive: the distribution has the longer tail on the upward side. Two distributions can share a mean and a variance and still differ here, which is the point the book makes at the start of Section 3.9: μ and σ alone do not pin down

$p(y)$.

Problem 3

seed 20260925

The operational-risk unit of a Baku bank tracks a weekly count Y (failed card transactions, disputed transfers, and so on) and, in its internal model, summarises Y only by its moment-generating function

$$m(t) = (0.40e^t + 0.60)^{19}.$$

(a) Which distribution does Y have?

- ○ Binomial
- ○ Geometric
- ○ Poisson
- ○ Hypergeometric

(b) By inspection, $E(Y) =$ ______________

(c) By inspection, $V(Y) =$ ______________

Give numerical answers to at least four significant figures.

(a) Exercise 3.145 shows that a binomial random variable with n trials and success probability p has $m(t) = (pe^t + q)^n$ with $q = 1 - p$. The MGF here has exactly that form with $n = 19$, $p = 0.40$ and $q = 0.60$ (and indeed $p + q = 1$). A moment-generating function determines its distribution uniquely, so Y is **binomial** with $n = 19$, $p = 0.40$.

(b) For a binomial random variable (Theorem 3.7) $E(Y) = np = 19(0.40) = 7.6000$.

(c) $V(Y) = npq = 19(0.40)(0.60) = 4.5600$.

Nothing needs differentiating: once the family and its parameters are identified, the moments come from the family's known formulas.

Problem 4

seed 20260925

A bank's call centre phones 14 pre-screened customers with a credit-card offer. Each customer accepts independently of the others, and Y, the number who accept, has moment-generating function

$$m(t) = (0.55e^t + 0.45)^{14}.$$

Differentiate $m(t)$ and use Theorem 3.12. Give answers to at least four significant figures.

(a) $E(Y) =$ ______________

(b) $E(Y^2) =$ ______________

(c) $V(Y) =$ ______________

Write $p = 0.55$, $q = 0.45$, $n = 14$, so $m(t) = (pe^t + q)^n$.

(a) By the chain rule,

$$m^{(1)}(t) = n(pe^t + q)^{n-1}\, pe^t.$$

At $t = 0$, $pe^0 + q = p + q = 1$, so by Theorem 3.12

$$E(Y) = m^{(1)}(0) = np = 14(0.55) = 7.7000.$$

(b) Differentiate the product $npe^t \cdot (pe^t + q)^{n-1}$ with the product rule:

$$m^{(2)}(t) = npe^t(pe^t + q)^{n-1} + n(n-1)p^2e^{2t}(pe^t + q)^{n-2}.$$

At $t = 0$ both brackets are 1:

$$E(Y^2) = m^{(2)}(0) = np + n(n-1)p^2 = 7.7000 + 14(13)(0.55)^2 = 7.7000 + 55.0550 = 62.7550.$$

(c) By Theorem 3.6,

$$V(Y) = E(Y^2) - [E(Y)]^2 = 62.7550 - (7.7000)^2 = 3.4650.$$

Algebraically, $np + n(n-1)p^2 - n^2p^2 = np - np^2 = npq$, and $npq = 3.4650$, which is the binomial variance of Theorem 3.7 recovered without summing a single series.

Problem 5

seed 20260925

Let Y be the number of business days it takes a cross-border payment sent through a Baku correspondent bank to settle. The bank's treasury summarises Y by its moment-generating function

$$m(t) = 0.24e^t + 0.33e^{2t} + 0.11e^{3t} + 0.32e^{4t}.$$

Give answers to at least four significant figures.

(a) $P(Y = 3) =$ ______________

(b) $P(Y \geq 3) =$ ______________

(c) $E(Y) =$ ______________

(d) $V(Y) =$ ______________

(a) For a discrete random variable, Definition 3.14 gives $m(t) = \sum_y e^{ty}p(y)$: each possible value y appears as an exponent e^{yt} with its probability as the coefficient. The random variable

y	1	2	3	4
$p(y)$	0.24	0.33	0.11	0.32

has exactly the MGF shown, and because a moment-generating function determines the distribution uniquely, this **is** the distribution of Y. So $P(Y = 3) = 0.11$, the coefficient of e^{3t}.

(b) $P(Y \geq 3) = p(3) + p(4) = 0.11 + 0.32 = 0.43$.

(c) Either sum directly or differentiate: $m^{(1)}(t) = 0.24e^{t} + 2(0.33)e^{2t} + 3(0.11)e^{3t} + 4(0.32)e^{4t}$, so

$$E(Y) = m^{(1)}(0) = 0.24 + 2(0.33) + 3(0.11) + 4(0.32) = 2.51.$$

(d) $m^{(2)}(0) = 0.24 + 4(0.33) + 9(0.11) + 16(0.32) = 7.67 = E(Y^2)$, and by Theorem 3.6

$$V(Y) = 7.67 - (2.51)^2 = 1.3699.$$

Problem 6

seed 20260925

A brokerage's risk system models Y, the number of margin calls a leveraged client receives in a month. Its documentation does not give $p(y)$; it gives only the first terms of the power-series expansion of the moment-generating function about $t = 0$:

$$m(t) = 1 + 1.11\,t + 1.17105\,t^2 + 1.028988\,t^3 + \cdots$$

Give answers to at least four significant figures.

(a) $\mu_1' = E(Y) =$ ______________

(b) $\mu_2' = E(Y^2) =$ ______________

(c) $V(Y) =$ ______________

(d) $\mu_3' = E(Y^3) =$ ______________

(e) Each margin call costs the client a 40 AZN handling fee. Expected monthly margin-call fees: ______________ AZN

Section 3.9 expands e^{ty} and takes expectations term by term:

$$m(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots,$$

so μ_k' is the coefficient of $t^k/k!$, **not** the coefficient of t^k. To read a moment off a series written in plain powers of t, multiply the coefficient of t^k by $k!$. (Equivalently, Theorem 3.12: differentiating $c_k t^k$ k times and setting $t = 0$ leaves $k!\,c_k$.)

(a) $\mu_1' = 1! \times 1.11 = 1.11$.

(b) $\mu_2' = 2! \times 1.17105 = 2.3421$.

(c) By Theorem 3.6, $V(Y) = \mu_2' - (\mu_1')^2 = 2.3421 - (1.11)^2 = 1.1100$.

(d) $\mu_3' = 3! \times 1.028988 = 6.1739$.

(e) Total fees are $40Y$, so by Theorem 3.4 the expected fees are $40\,E(Y) = 40 \times 1.11 = 44.40$ AZN.

The mean and the variance come out equal here. That is a hint, not a proof, that Y might be Poisson with $\lambda = 1.11$; the whole series $e^{1.11(e^t-1)}$ would have to match for the uniqueness argument of Example 3.25 to apply, and three terms do not establish that.

Problem 7

seed 20260925

A Baku dealer makes a market in a thinly traded corporate bond. The number of trades Y it executes in the bond on a given day has moment-generating function

$$m(t) = e^{1.60(e^t-1)}.$$

Give answers to at least four significant figures.

(a) Use Theorem 3.12 to find the third moment about the origin, $\mu_3' = E(Y^3) =$ ______________

(b) Find the third central moment $\mu_3 = E[(Y-\mu)^3] =$ ______________

(c) The dealer earns a flat 9 AZN commission per trade, so its daily commission revenue is $R = 9Y$. Find the third central moment of R, $E[(R - E(R))^3] =$ ______________

Write $\lambda = 1.60$ and $m(t) = e^{\lambda(e^t-1)}$. Note that $\frac{d}{dt}\lambda(e^t-1) = \lambda e^t$, so $m^{(1)}(t) = m(t)\,\lambda e^t$.

(a) Differentiate repeatedly with the product rule, each time using $m^{(1)} = m\lambda e^t$:

$$m^{(1)}(t) = m(t)\,\lambda e^t,$$
$$m^{(2)}(t) = m(t)\,\left[(\lambda e^t)^2 + \lambda e^t\right],$$
$$m^{(3)}(t) = m(t)\,\left[(\lambda e^t)^3 + 3(\lambda e^t)^2 + \lambda e^t\right].$$

(For the last line: the derivative of the bracket is $2(\lambda e^t)^2 + \lambda e^t$, and adding it to λe^t times the bracket gives the terms shown.) At $t = 0$, $m(0) = 1$ and $\lambda e^0 = \lambda$, so by Theorem 3.12

$$\mu_1' = \lambda, \qquad \mu_2' = \lambda^2 + \lambda = 4.1600,$$
$$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda = 4.0960 + 7.6800 + 1.60 = 13.3760.$$

(b) Expanding $(Y-\mu)^3$ and using Theorem 3.5,

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = 13.3760 - 19.9680 + 8.1920 = 1.6000.$$

In symbols, $\lambda^3 + 3\lambda^2 + \lambda - 3\lambda(\lambda^2+\lambda) + 2\lambda^3 = \lambda$: a Poisson count has its mean, its variance **and** its third central moment all equal to λ. The positive third moment reflects the long right tail: on a quiet day the dealer cannot trade fewer than zero times, but on a busy day it can trade many more than λ.

(c) $R - E(R) = 9Y - 9\mu = 9(Y-\mu)$, so

$$E[(R-E(R))^3] = 9^3\,E[(Y-\mu)^3] = 729 \times 1.6000 = 1166.4000.$$

Scaling by c scales the kth central moment by c^k. (Equivalently, $m_R(t) = E(e^{t\,9Y}) = m(9t)$, and each derivative of $m(9t)$ brings out another factor 9.)

Problem 8

seed 20260925

A proprietary trader places 8 short-horizon bets on unrelated events. Each bet wins with probability 0.66, independently of the others. A winning bet gains 2 and a losing bet loses 4, both in hundreds of AZN. Let Y be the number of winning bets, so Y is binomial with $m_Y(t) = (0.66e^t + 0.34)^8$, and let X be the trader's total P&L (hundreds of AZN).

(a) Express X as $X = aY + b$ and use it to write the moment-generating function of X. (Type e^{2t} as e^(2*t).)

$m_X(t) =$ ______________

(b) $E(X) =$ ______________ (hundreds of AZN)

(c) $V(X) =$ ______________ (squared hundreds of AZN)

Give numerical answers to at least four significant figures.

(a) With Y wins there are $8 - Y$ losses, so

$$X = 2Y - 4(8 - Y) = 6Y - 32.$$

For $X = aY + b$, $m_X(t) = E(e^{t(aY+b)}) = e^{bt}E(e^{(at)Y}) = e^{bt}m_Y(at)$ (this is Exercise 3.156(b) and (c) combined). With $a = 6, b = -32$:

$$m_X(t) = e^{-32t}\left(0.66e^{6t} + 0.34\right)^8.$$

(b) Differentiate $m_X(t) = e^{bt}m_Y(at)$ by the product and chain rules: $m_X^{(1)}(t) = be^{bt}m_Y(at) + ae^{bt}m_Y^{(1)}(at)$. At $t = 0$, $m_Y(0) = 1$ and $m_Y^{(1)}(0) = E(Y) = np$, so

$$E(X) = m_X^{(1)}(0) = b + a\,np = -32 + 6(5.28) = -0.32.$$

(c) One more derivative: $m_X^{(2)}(0) = b^2 + 2ab\,E(Y) + a^2E(Y^2)$, and subtracting $[E(X)]^2 = b^2 + 2ab\,E(Y) + a^2[E(Y)]^2$ leaves

$$V(X) = a^2\left(E(Y^2) - [E(Y)]^2\right) = a^2V(Y) = 36 \times npq = 36 \times 1.7952 = 64.6272.$$

The shift $b = -32$ (the loss if every bet failed) moves the mean but contributes only the factor e^{bt} to the MGF, which has no effect on the variance. The spread is set entirely by $a = 6$, the swing between a win and a loss.

Problem 9

seed 20260925

A motor insurer in Baku notices that many of its policies cover cars that are hardly ever driven. Its actuaries model the number of claims Y on a randomly chosen policy in a year through the moment-generating function

$$m(t) = 0.20 + 0.80\,e^{1.40(e^t - 1)}.$$

Give answers to at least four significant figures.

(a) $P(Y=0) =$ ______________

(b) $E(Y) =$ ______________

(c) $E(Y^2) =$ ______________

(d) $V(Y) =$ ______________

Write $w = 0.20$ and $\lambda = 1.40$, so $m(t) = w + (1-w)e^{\lambda(e^t-1)}$.

(a) By Definition 3.14, $m(t) = \sum_y e^{ty}p(y)$, so $p(0)$ is the coefficient of $e^{0\cdot t}$, the part of $m(t)$ that does not depend on t once everything is written as a sum of terms $c_y e^{yt}$. From Example 3.23,

$$e^{\lambda(e^t-1)} = \sum_{y=0}^{\infty} e^{ty}\,\frac{\lambda^y e^{-\lambda}}{y!},$$

whose $y=0$ term is $e^{-\lambda}$. So

$$m(t) = \left[w + (1-w)e^{-\lambda}\right] + \sum_{y\geq 1} e^{ty}(1-w)\frac{\lambda^y e^{-\lambda}}{y!},$$

and by uniqueness $P(Y=0) = w + (1-w)e^{-\lambda} = 0.20 + 0.80(0.246597) = 0.3973$.

The model says a policy is claim-free for two reasons: it belongs to the “never driven” group (probability w), or it is an ordinary policy whose Poisson count happens to be zero.

(b) The constant w vanishes on differentiation, and by Example 3.24 the derivatives of $e^{\lambda(e^t-1)}$ at 0 are λ and $\lambda^2+\lambda$. So

$$E(Y) = m^{(1)}(0) = (1-w)\lambda = 0.80(1.40) = 1.1200.$$

(c) $E(Y^2) = m^{(2)}(0) = (1-w)(\lambda^2+\lambda) = 0.80(3.3600) = 2.6880$.

(d) By Theorem 3.6, $V(Y) = 2.6880 - (1.1200)^2 = 1.4336$.

This exceeds the mean 1.1200: in symbols $V(Y) - E(Y) = w(1-w)\lambda^2 > 0$. A plain Poisson count has variance equal to its mean, so a claims history whose variance is visibly larger than its mean is one piece of evidence that a single Poisson model is not enough.

Problem 10

seed 20260925

An agent for a Baku life insurer phones prospects one at a time. Calls are independent and each ends in a sale with the same probability. Let Y be the number of the call on which the agent makes the first sale of the day. The insurer’s model gives Y the moment-generating function

$$m(t) = \frac{0.14e^t}{1-0.86e^t}.$$

Give answers to at least four significant figures.

(a) Use Theorem 3.12: $E(Y) =$ ______________

(b) $E(Y^2) =$ ______________

(c) $V(Y) =$ ______________

(d) Each call costs the insurer 12 AZN in the agent's time and phone charges. Expected cost of the calls up to and including the first sale: ______________ AZN

(e) $P(Y \le 3) =$ ______________

Write $p = 0.14$, $q = 0.86$. By Exercise 3.147, this is the MGF of a geometric random variable with success probability $p = 0.14$.

(a) By the quotient rule,

$$m^{(1)}(t) = \frac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}.$$

At $t = 0$, $1 - q = p$, so $E(Y) = m^{(1)}(0) = p/p^2 = 1/p = 1/0.14 = 7.1429$.

(b) Differentiate again:

$$m^{(2)}(t) = \frac{pe^t(1-qe^t)^2 - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}.$$

At $t = 0$: $E(Y^2) = m^{(2)}(0) = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2} = \frac{1.86}{0.14^2} = 94.8980$.

(c) By Theorem 3.6,

$$V(Y) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} = \frac{0.86}{0.14^2} = 43.8776,$$

the geometric variance of Theorem 3.8.

(d) The cost is $12Y$, so by Theorem 3.4 its expectation is $12E(Y) = 12/0.14 = 85.71$ AZN.

(e) $Y > 3$ means the first 3 calls all fail, which has probability q^3. So $P(Y \le 3) = 1 - 0.86^3 = 1 - 0.636056 = 0.3639$.

Problem 11

seed 20260925

A bank has 10 branches. Each branch meets its monthly deposit-growth target with probability 0.84, independently of the others, so the number Y of branches that meet it is binomial with $m_Y(t) = (0.84e^t + 0.16)^{10}$ and mean $\mu = 8.40$.

(a) Write the moment-generating function of $X = Y - \mu$, the deviation of the count from its mean. (Type e^{2t} as e^(2*t).)

$m_X(t) =$ ______________

(b) Theorem 3.12 applied to m_X gives the central moments of Y. Find $\mu_2 = E[(Y-\mu)^2] =$ ______________

(c) $\mu_3 = E[(Y-\mu)^3] =$ ______________

(d) Which describes the distribution of Y?

- ◯ Longer right tail (third central moment positive)
- ◯ Longer left tail (third central moment negative)
- ◯ Symmetric about the mean (third central moment zero)

Give numerical answers to at least four significant figures.

(a) $m_X(t) = E(e^{t(Y-\mu)}) = e^{-\mu t}E(e^{tY}) = e^{-\mu t}m_Y(t)$ (Exercise 3.156(c) with the shift μ), so

$$m_X(t) = e^{-8.40t}\left(0.84e^t + 0.16\right)^{10}.$$

The moments of X about the origin are, by Definition 3.13, the central moments of Y: $E(X^k) = E[(Y-\mu)^k] = \mu_k$.

(b) Multiplying out is messy; it is quicker to write $m_X(t) = \left(pe^t + q\right)^n e^{-npt} = \left[e^{-pt}(pe^t + q)\right]^n = \left(pe^{qt} + qe^{-pt}\right)^n$, which is the nth power of the MGF of one centred Bernoulli trial $B - p$ (it equals q with probability p and $-p$ with probability q). Write $h(t) = pe^{qt} + qe^{-pt}$. Then $h(0) = 1$, $h^{(1)}(0) = pq - qp = 0$, $h^{(2)}(0) = pq^2 + qp^2 = pq$, $h^{(3)}(0) = pq^3 - qp^3 = pq(q^2 - p^2) = pq(q-p)$.

For $m_X = h^n$: $m_X^{(1)} = nh^{n-1}h'$, $m_X^{(2)} = n(n-1)h^{n-2}(h')^2 + nh^{n-1}h''$, and at $t = 0$, where $h' = 0$,

$$\mu_2 = m_X^{(2)}(0) = n\,h^{(2)}(0) = npq = 10(0.84)(0.16) = 1.3440.$$

(c) Differentiating once more, every term of $m_X^{(3)}$ carries a factor h' except $nh^{n-1}h'''$, so

$$\mu_3 = m_X^{(3)}(0) = n\,h^{(3)}(0) = npq(q-p) = 1.3440 \times (-0.68) = -0.9139.$$

(The same number comes from $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$ with the raw moments read off m_Y, but the centred MGF avoids the cancellation.)

(d) The sign of μ_3 is the sign of $q - p$. Here $p = 0.84$, so the correct choice is: Longer left tail (third central moment negative). When success is rare the count is bunched near zero with a tail to the right; when success is common it is bunched near n with a tail to the left; at $p = 0.5$ the binomial is symmetric.

Problem 12

seed 20260925

A fintech start-up's fraud model produces a count Y each hour. The model's documentation gives only the moment-generating function, written by a developer who did not simplify it:

$$m(t) = \frac{(5e^t + 2)^{11}}{7^{11}}.$$

(a) Which distribution does Y have?

- ◯ Binomial
- ◯ Geometric
- ◯ Poisson
- ◯ None of these

(b) $E(Y) =$ ______________

(c) $V(Y) =$ ______________

Give numerical answers to at least four significant figures.

(a) Divide each factor of the numerator by 7:

$$m(t) = \left(\frac{5e^t + 2}{7}\right)^{11} = \left(\frac{5}{7}e^t + \frac{2}{7}\right)^{11}.$$

This is $(pe^t + q)^n$ with $p = 5/7 = 0.7143$, $q = 2/7 = 0.2857$ (note $p + q = 1$) and $n = 11$, the binomial MGF of Exercise 3.145. Since a moment-generating function determines the distribution, Y is **binomial** with $n = 11$, $p = 5/7$.

(b) $E(Y) = np = 11(5/7) = 7.8571$ (Theorem 3.7).

(c) $V(Y) = npq = 11(5/7)(2/7) = 2.2449$.

A useful check before accepting any candidate: $m(0)$ must equal 1. Here $m(0) = 7^{11}/7^{11} = 1$.

Problem 13

seed 20260925

A mortgage broker in Baku submits 10 applications this week. A model the broker's analyst fitted says the number Y of those applications that are approved has moment-generating function

$$m(t) = (0.76e^t + 0.24)^{10}.$$

Give every probability to at least four significant figures (enter decimals).

(a) $P(Y = 7) =$ ______________

(b) $P(Y \leq 7) =$ ______________

By Exercise 3.145 the binomial distribution with n trials and success probability p has MGF $(pe^t + q)^n$. The MGF here is of that form with $n = 10$, $p = 0.76$, $q = 0.24$, and a moment-generating function determines the distribution uniquely. So Y is binomial with $n = 10$, $p = 0.76$, and its probabilities come from Definition 3.7:

$$p(y) = \binom{10}{y}(0.76)^y(0.24)^{10-y}, \qquad y = 0, 1, \ldots, 10.$$

(a) $\binom{10}{7} = 120$, so

$$P(Y=7) = 120\,(0.76)^7(0.24)^3 = 0.242946.$$

(b) Add the terms for $y = 0, 1, \ldots, 7$:

$$P(Y \le 7) = \sum_{y=0}^{7} \binom{10}{y} (0.76)^y (0.24)^{10-y} = 0.444195.$$

A quick way to generate the terms is the ratio $p(y)/p(y-1) = \frac{10-y+1}{y} \cdot \frac{0.76}{0.24}$, starting from $p(0) = 0.24^{10}$. For reference, the mean is $np = 7.60$, so 7 approvals is close to what the model expects.

The MGF did not have to be expanded or differentiated here. Its only job was to name the distribution, which is its second (and, the book says, primary) use.

Problem 14

seed 20260925

The card-fraud desk of a Baku bank receives Y automated alerts per hour during the evening shift. From a year of logs, the desk models Y with moment-generating function

$$m(t) = e^{4.38(e^t-1)}.$$

Give answers to at least four significant figures.

(a) $\mu = E(Y) =$ ________________

(b) $\sigma =$ ________________

(c) The desk staffs for “normal” hours, those with $|Y-\mu| \le 2\sigma$. Find $P(|Y-\mu| \le 2\sigma) =$ ________________

(a) The MGF is $e^{\lambda(e^t-1)}$ with $\lambda = 4.38$, which by Example 3.23 is the Poisson MGF. A moment-generating function determines the distribution (Example 3.25), so Y is Poisson with mean $\mu = \lambda = 4.38$.

(b) From Example 3.24, $\sigma^2 = \lambda$, so $\sigma = \sqrt{4.38} = 2.0928$.

(c) The event $|Y-\mu| \le 2\sigma$ is

$$4.38 - 2(2.0928) \le Y \le 4.38 + 2(2.0928), \quad \text{i.e.} \quad 0.1943 \le Y \le 8.5657.$$

Y takes only integer values, so this is $1 \le Y \le 8$, and

$$P(|Y-\mu| \le 2\sigma) = \sum_{y=1}^{8} \frac{4.38^y e^{-4.38}}{y!} = 0.9525.$$

Generate the terms with the ratio $p(y) = p(y-1) \cdot 4.38/y$ starting from $p(0) = e^{-4.38}$ rather than evaluating factorials. Rounding the endpoints to the integers inside the interval is the step that is easy to get wrong: the endpoints themselves are usually not attainable values of Y.

Problem 15

seed 20260925

A corporate treasury in Baku holds a one-week currency position. Its P&L X, in thousands of US dollars, is modelled as

x	-4	0	6
$p(x)$	0.20	0.37	0.43

(Type e^{2t} as e^(2*t).)

(a) $m_X(t) =$ ______________

(b) Use Theorem 3.12: $E(X) =$ ______________ (thousand USD)

(c) $V(X) =$ ______________

(d) The treasury reports in manat, at 1.7 AZN per USD, so the P&L in thousands of AZN is $W = 1.7X$. Write the moment-generating function of W.

$m_W(t) =$ ______________

(e) $V(W) =$ ______________

Give numerical answers to at least four significant figures.

(a) By Definition 3.14, $m_X(t) = \sum_x e^{tx} p(x)$, and a negative value simply gives a negative exponent:

$$m_X(t) = 0.20e^{-4t} + 0.37 + 0.43e^{6t}.$$

(b) $m_X^{(1)}(t) = -4(0.20)e^{-4t} + 6(0.43)e^{6t}$, so

$$E(X) = m_X^{(1)}(0) = -4(0.20) + 6(0.43) = 1.78.$$

(c) $m_X^{(2)}(t) = 16(0.20)e^{-4t} + 36(0.43)e^{6t}$ (the sign of -4 is squared away), so $E(X^2) = m_X^{(2)}(0) = 18.68$ and by Theorem 3.6

$$V(X) = 18.68 - (1.78)^2 = 15.5116.$$

(d) $m_W(t) = E(e^{t(1.7X)}) = E(e^{(1.7t)X}) = m_X(1.7t)$ (Exercise 3.156(b) with 1.7 in place of 3). Replace t by $1.7t$ everywhere:

$$m_W(t) = 0.20e^{-6.8t} + 0.37 + 0.43e^{10.2t}.$$

This is also what Definition 3.14 gives directly: W takes the values -6.8, 0, 10.2 with the same probabilities.

(e) By the chain rule, $m_W^{(k)}(t) = 1.7^k m_X^{(k)}(1.7t)$, so $E(W) = 1.7E(X)$, $E(W^2) = 1.7^2 E(X^2)$, and

$$V(W) = 1.7^2 \left(E(X^2) - [E(X)]^2\right) = 2.89 \times 15.5116 = 44.8285.$$

(Exercise 3.157(a) is the same argument with the factor 3.) Currency conversion rescales the standard deviation by the rate and the variance by its square.

Problem 16

seed 20260925

A small Baku business applies for a working-capital loan at one bank after another, stopping at the first approval. Decisions are independent and each bank approves with the same probability. The number Y of the application on which the first approval comes has moment-generating function

$$m_Y(t) = \frac{0.22e^t}{1 - 0.78e^t}.$$

Let $X = Y - 1$ be the number of **rejections** before the first approval.

(a) Write the moment-generating function of X and simplify it.

$m_X(t) =$ ______________

(b) Use Theorem 3.12 on m_X: $E(X) =$ ______________

(c) $V(X) =$ ______________

(d) Every rejected application costs the business a non-refundable 75 AZN appraisal fee. Expected total of such fees: ______________ AZN

(e) $P(X \geq 3) =$ ______________

Give numerical answers to at least four significant figures.

Write $p = 0.22$, $q = 0.78$.

(a) Exercise 3.156(c) with the shift 1: $m_X(t) = E(e^{t(Y-1)}) = e^{-t}m_Y(t)$, and the e^t in the numerator cancels:

$$m_X(t) = e^{-t}\frac{pe^t}{1 - qe^t} = \frac{0.22}{1 - 0.78e^t}.$$

(b) $m_X^{(1)}(t) = \frac{pqe^t}{(1-qe^t)^2}$, so $E(X) = m_X^{(1)}(0) = \frac{pq}{p^2} = \frac{q}{p} = \frac{0.78}{0.22} = 3.5455.$

As Exercise 3.157(b) says it must, this is $E(Y) - 1 = 1/p - 1 = 4.5455 - 1.$

(c) Differentiate again with the quotient rule:

$$m_X^{(2)}(t) = \frac{pqe^t(1 - qe^t)^2 + pqe^t \cdot 2(1 - qe^t)qe^t}{(1 - qe^t)^4} = \frac{pqe^t(1 + qe^t)}{(1 - qe^t)^3},$$

so $E(X^2) = m_X^{(2)}(0) = q(1+q)/p^2$, and by Theorem 3.6

$$V(X) = \frac{q(1+q)}{p^2} - \frac{q^2}{p^2} = \frac{q}{p^2} = \frac{0.78}{0.22^2} = 16.1157.$$

This equals $V(Y)$ (Theorem 3.8): shifting by a constant multiplies the MGF by e^{-t} and leaves the variance alone.

(d) Fees are $75X$, so by Theorem 3.4 the expected fees are $75 \times 3.5455 = 265.91$ AZN.

(e) $X \geq 3$ means the first 3 banks all reject, so $P(X \geq 3) = q^3 = 0.78^3 = 0.4746$.

Problem 17

seed 20260925

A market-maker's risk report gives only the moment-generating function of its P&L X over one trading session, in hundreds of AZN:

$$m_X(t) = e^{-20t}\left(0.45e^{2t} + 0.55\right)^{15}.$$

Give answers to at least four significant figures.

(a) $E(X) =$ ______________

(b) $V(X) =$ ______________

(c) $P(X > 0) =$ ______________

(Hint: find a random variable Y you recognise and constants with $X = aY + b$.)

(a) If Y is binomial with $n = 15, p = 0.45$ (Exercise 3.145), then $m_Y(t) = (0.45e^t + 0.55)^{15}$, and the random variable $2Y - 20$ has MGF

$$E(e^{t(2Y-20)}) = e^{-20t}E(e^{(2t)Y}) = e^{-20t}m_Y(2t) = e^{-20t}\left(0.45e^{2t} + 0.55\right)^{15},$$

which is exactly $m_X(t)$. Moment-generating functions determine distributions, so X has the distribution of $2Y - 20$: think of 15 independent quotes, each earning 2 (hundred AZN) with probability 0.45, against a fixed session cost of 20.

Then $E(X) = 2E(Y) - 20 = 2(6.75) - 20 = -6.50$. (Differentiating m_X once and setting $t = 0$ gives the same: $m_X^{(1)}(0) = -20 + 2\,m_Y^{(1)}(0)$.)

(b) $V(X) = 2^2V(Y) = 4 \times npq = 4 \times 3.7125 = 14.8500$. The constant -20 contributes only the factor e^{-20t}, which moves the mean and not the spread.

(c) $X > 0 \iff 2Y > 20 \iff Y > 20/2 = 10.0000$. Since Y is an integer this is $Y \geq 11$, so

$$P(X > 0) = \sum_{y=11}^{15} \binom{15}{y} (0.45)^y (0.55)^{15-y} = 0.0255.$$

Watch the strict inequality: whenever b/a happens to be a whole number, a session with exactly b/a profitable quotes breaks even, $X = 0$, and that is not a profit.

Problem 18

seed 20260925

A Baku bank buys protection on a corporate borrower. The borrower survives each year independently with probability 0.96 and defaults with probability 0.04, so the year Y in which it first defaults ($Y = 1, 2, 3, \ldots$) is geometric with $p = 0.04$. If default happens, the protection seller pays 5 million AZN at the end of year Y. With continuously compounded interest at $r = 0.115$, the value today of that payment is $5D$, where $D = e^{-rY}$ is the discount factor.

(a) Starting from Definition 3.14, derive the moment-generating function of Y.

$m(t) =$ ______________

(b) Explain to yourself why $E(D) = m(-r)$, then compute $E(D) =$ ______________

(c) Expected present value of the protection payment: ______________ million AZN

(d) $V(D) =$ ______________

Give numerical answers to at least four significant figures.

(a) For the geometric distribution $p(y) = q^{y-1}p$, $y = 1, 2, \ldots$ (Definition 3.8). So

$$m(t) = \sum_{y=1}^{\infty} e^{ty} q^{y-1} p = pe^t \sum_{y=1}^{\infty} (qe^t)^{y-1} = \frac{pe^t}{1 - qe^t},$$

a geometric series that converges when $qe^t < 1$, i.e. $t < -\ln q$. With the numbers,

$$m(t) = \frac{0.04e^t}{1 - 0.96e^t}.$$

(b) $E(D) = E(e^{-rY}) = E(e^{tY})\big|_{t=-r} = m(-r)$. The moment-generating function is an ordinary function of t; Theorem 3.12 uses its derivatives at 0, but its **value** at $t = -r$ is exactly an expected discount factor. Negative t is always inside the region of convergence here.

$$E(D) = m(-0.115) = \frac{0.04e^{-0.115}}{1 - 0.96e^{-0.115}} = \frac{0.04(0.891366)}{1 - 0.96(0.891366)} = 0.247107.$$

(c) By Theorem 3.4, $E(5D) = 5E(D) = 5 \times 0.247107 = 1.2355$ million AZN. This is the fair up-front price of the protection leg, before any premium payments.

(d) $D^2 = e^{-2rY}$, so $E(D^2) = m(-2r) = m(-0.230)$:

$$E(D^2) = \frac{0.04(0.794534)}{1 - 0.96(0.794534)} = 0.133958,$$

and by Theorem 3.6

$$V(D) = m(-2r) - [m(-r)]^2 = 0.133958 - (0.247107)^2 = 0.072897.$$

The same idea values any payment that depends on a random date: evaluate the MGF of the date at minus the interest rate.

Problem 19

seed 20260925

A share listed on the Baku Stock Exchange trades at 32 AZN today. An analyst models next month's price as

$$S = 32\,(0.93)^Y,$$

where Y, the number of negative news shocks in the month, is Poisson with $m_Y(t) = e^{3.8(e^t-1)}$, and each shock knocks 7% off the price. (Nothing else moves the price in this stylised model.)

(a) $S = 32e^{t^*Y}$ for a constant t^*. Find $t^* =$ ____________

(b) $E(S) =$ ____________ AZN

(c) $V(S) =$ ____________ (AZN squared)

Give answers to at least four significant figures.

(a) $(0.93)^Y = e^{Y\ln(0.93)}$, so $t^* = \ln(0.93) = -0.072571$.

(b) By Definition 3.14, $E(e^{t^*Y}) = m_Y(t^*)$: the expectation we want is the MGF evaluated at the (negative) point t^*, not a derivative at 0. So

$$E(S) = 32\,m_Y(t^*) = 32\,e^{\lambda(e^{t^*}-1)} = 32\,e^{\lambda(1-d-1)} = 32\,e^{-\lambda d},$$

$$E(S) = 32\,e^{-3.8(0.07)} = 32\,e^{-0.2660} = 24.5261 \text{ AZN}.$$

(c) $S^2 = 32^2e^{2t^*Y}$, so $E(S^2) = 1024\,m_Y(2t^*)$ with $e^{2t^*} = (0.93)^2 = 0.8649$:

$$E(S^2) = 1024\,e^{3.8(0.8649-1)} = 1024\,e^{3.8(-0.1351)} = 612.8326.$$

By Theorem 3.6,

$$V(S) = E(S^2) - [E(S)]^2 = 612.8326 - (24.5261)^2 = 11.3054,$$

a standard deviation of 3.3623 AZN.

Note that $E(S) = 32e^{-\lambda d}$ is not $32(1-d)^\lambda$, the price after the **expected** number of shocks. The two differ because $E[g(Y)] \neq g(E(Y))$ for a non-linear g; the MGF is the tool that computes $E(e^{tY})$ exactly.

Problem 20

seed 20260925

A credit analyst studies the loss L (in thousands of AZN) that a bank takes on one small-business loan over a year. She does not have $p(\ell)$, but a vendor model hands her every moment about the origin:

$$\mu'_k = E(L^k) = 0.07 \times 89^k, \qquad k = 1, 2, 3, \ldots$$

Assume the MGF of L exists.

(a) Use the series $m(t) = 1 + \sum_{k\geq 1} \mu'_k\, t^k/k!$ to find the moment-generating function in closed form. (Type e^{2t} as e^(2*t).)

$m(t) =$ ______________

(b) Identify the distribution of L from (a). $P(L = 0) =$ ______________

(c) $V(L) =$ ______________

(d) The third central moment $\mu_3 = E[(L-\mu)^3] =$ ______________

Give numerical answers to at least four significant figures.

Write $\pi = 0.07$ and $a = 89$.

(a) Section 3.9 shows $m(t) = 1 + t\mu'_1 + \frac{t^2}{2!}\mu'_2 + \cdots$. Substituting $\mu'_k = \pi a^k$,

$$m(t) = 1 + \pi \sum_{k=1}^{\infty} \frac{(at)^k}{k!} = 1 + \pi\left(e^{at} - 1\right) = 0.93 + 0.07e^{89t},$$

using $\sum_{k\geq 0} x^k/k! = e^x$ with the $k = 0$ term (which is 1) removed.

(b) By Definition 3.14 a random variable taking the value 0 with probability $1 - \pi$ and a with probability π has MGF $(1-\pi)e^{0\cdot t} + \pi e^{at}$, which is the function in (a). Since an MGF determines the distribution, L is that two-point random variable: the bank loses nothing with probability 0.93 and loses 89 thousand AZN (default) with probability 0.07. So $P(L = 0) = 0.93$.

This is the argument the book sketches at the start of Section 3.9: under mild conditions the full sequence of moments pins down the distribution.

(c) $\mu = \mu'_1 = 0.07(89) = 6.23$ and $\mu'_2 = \pi a^2 = 554.47$, so

$$V(L) = \pi a^2 - \pi^2 a^2 = a^2\pi(1-\pi) = 7921(0.07)(0.93) = 515.6571.$$

(d) Using $\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3$ with $\mu'_3 = \pi a^3 = 49347.83$:

$$\mu_3 = a^3\left(\pi - 3\pi^2 + 2\pi^3\right) = a^3\pi(1-\pi)(1-2\pi) = 704969(0.07)(0.93)(0.86) = 39468.39.$$

With $\pi < 0.5$ this is positive: the loss distribution has a long right tail (usually nothing, occasionally a large loss), the typical shape of credit losses and the reason banks look beyond the mean and variance.

Answer key

Problem	Answers
1	(a) 0.47+0.24*e^t+0.24*e^(2*t)+0.05*e^(3*t), (b) 0.87, (c) 1.65
2	(a) 0.36, (b) 0.78, (c) 0.84, (d) 0.6504, (e) 0.090912
3	(a) Binomial, (b) 7.6, (c) 4.56

4	(a) 7.7, (b) 62.755, (c) 3.465
5	(a) 0.11, (b) 0.43, (c) 2.51, (d) 1.3699
6	(a) 1.11, (b) 2.3421, (c) 1.11, (d) 6.17393, (e) 44.4
7	(a) 13.376, (b) 1.6, (c) 1166.4
8	(a) e^(-32*t)*[0.66*e^(6*t)+0.34]^8, (b) -0.32, (c) 64.6272
9	(a) 0.397278, (b) 1.12, (c) 2.688, (d) 1.4336
10	(a) 7.14286, (b) 94.898, (c) 43.8776, (d) 85.7143, (e) 0.363944
11	(a) e^(-8.4*t)*(0.84*e^t+0.16)^10, (b) 1.344, (c) -0.91392, (d) Longer left ... negative)
12	(a) Binomial, (b) 7.85714, (c) 2.2449
13	(a) 0.242946, (b) 0.444195
14	(a) 4.38, (b) 2.09284, (c) 0.95252
15	(a) 0.2*e^(-4*t)+0.37+0.43*e^(6*t), (b) 1.78, (c) 15.5116, (d) 0.2*e^(-6.8*t)+0.37+0.43*e^(10.2*t), (e) 44.8285
16	(a) 0.22/(1-0.78*e^t), (b) 3.54545, (c) 16.1157, (d) 265.909, (e) 0.474552
17	(a) -6.5, (b) 14.85, (c) 0.0254659
18	(a) 0.04*e^t/(1-0.96*e^t), (b) 0.247107, (c) 1.23553, (d) 0.0728968
19	(a) -0.0725707, (b) 24.5261, (c) 11.3054
20	(a) 0.93+0.07*e^(89*t), (b) 0.93, (c) 515.657, (d) 39468.4