Week 8, Problem Set 2

Wackerly 3.10-3.11 - Probability-Generating Functions, Tchebysheff's Theorem, and Chapter 3 in Review

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A bank's Baku branch network has recorded, over several years, the weekly number Y of mortgage applications it receives. The long-run mean is $\mu = 156$ and the standard deviation is $\sigma = 24$; nothing else is known about the shape of the distribution.

The mortgage desk wants to know how likely it is that next week's count falls strictly between 96 and 216.

Give every probability to at least four significant figures.

(a) Writing the interval as $(\mu - k\sigma,\ \mu + k\sigma)$, the value of k is _____

(b) The best lower bound that Tchebysheff's theorem gives for $P(96 < Y < 216)$ is _____

(c) The corresponding upper bound for the probability that next week's count is 96 or fewer, or 216 or more, is _____

(d) Suppose instead that the standard deviation were only $\sigma = 12$ (same mean). The Tchebysheff lower bound for $P(96 < Y < 216)$ would then be _____

Problem 2

seed 20260925

An insurance agency in Ganja records Y, the number of motor-insurance claims it receives in a month. From past data $E(Y) = 64$ and $V(Y) = 144$. The shape of the distribution is not known.

Give every answer to at least four significant figures.

(a) Using Tchebysheff's theorem, a lower bound for $P(46 < Y < 82)$ is _____

(b) The agency wants a margin C such that, whatever the distribution, $P(|Y - 64| \geq C) \leq 0.18$. The smallest C that Tchebysheff's theorem guarantees this for is $C =$ _____

(c) The upper end of the interval $\mu \pm C$ found in (b) is _____

Problem 3

seed 20260925

A microfinance lender in Lankaran tracks Y, the number of late instalments a borrower makes on a small business loan over one quarter (three instalments). Across its book the distribution is

y	0	1	2	3
$p(y)$	0.71	0.13	0.14	0.02

Give every number to at least four significant figures.

(a) The probability-generating function of Y is $P(t) =$ _____

(Enter a function of t, for example 0.5 + 0.3*t + 0.2*t^2.)

(b) The first factorial moment $\mu_{[1]} = P^{(1)}(1) =$ _____

(c) The second factorial moment $\mu_{[2]} = E[Y(Y-1)] =$ _____

(d) $V(Y) =$ _____

(e) The third factorial moment $\mu_{[3]} = E[Y(Y-1)(Y-2)] =$ _____

Problem 4

seed 20260925

A bank has just issued 38 small-business loans to unrelated firms in different regions. Each loan, independently of the others, defaults within a year with probability 0.15. Let Y be the number of these loans that default within a year.

Give every number to at least four significant figures.

(a) Which probability distribution does Y have?

- ○ Binomial
- ○ Geometric
- ○ Negative binomial
- ○ Hypergeometric
- ○ Poisson

(b) $E(Y) =$ _____

(c) $V(Y) =$ _____

Problem 5

seed 20260925

A bank's call centre makes 160 outbound calls offering a new credit card. Each call reaches a different customer, and each customer independently accepts the offer with probability 0.28. Let Y be the number of acceptances.

Give every number to at least four significant figures.

(a) $\mu = E(Y) =$ _____

(b) $\sigma =$ _____

(c) The interval $\mu \pm 2\sigma$ runs from _____ to _____. (By Tchebysheff's theorem it contains Y with probability at least $3/4$.)

(d) The product team says the campaign is a success only if at least 65 customers accept. Use Tchebysheff's theorem to give an upper bound for $P(Y \geq 65)$: _____

Problem 6

seed 20260925

The number Y of cross-border payment messages that fail validation at a Baku bank in one day has a Poisson distribution with mean $\lambda = 3.18$.

Give every number to at least four significant figures.

(a) The probability-generating function of Y is $P(t) =$ _____

(Enter a function of t; e^(...) and exp(...) both work.)

(b) $\mu_{[1]} = P^{(1)}(1) =$ _____

(c) $\mu_{[2]} = P^{(2)}(1) =$ _____

(d) $V(Y) =$ _____

(e) Each failed message is repaired automatically, independently of the others, with probability 0.96; the rest need a person. The probability that **every** failed message today is repaired automatically (which includes the case of no failures at all) equals $P(0.96)$. Find it: _____

Problem 7

seed 20260925

A Baku brokerage opens, on average, $\mu = 200$ new retail trading accounts per business day, with standard deviation $\sigma = 14$. The daily count is clearly not bell-shaped (it jumps on days with IPO news), so the compliance team will rely only on Tchebysheff's theorem.

Give every number to at least four significant figures.

(a) The smallest k for which Tchebysheff's theorem guarantees $P(|Y - \mu| < k\sigma) \geq 0.88$ is $k =$ _____

(b) The corresponding interval $(\mu - k\sigma,\ \mu + k\sigma)$, which contains the daily count with probability at least 0.88, has lower end _____ and upper end _____

(c) The team also wants a half-width w with $P(|Y - \mu| \geq w) \leq 0.09$ guaranteed. The smallest such w is _____

Problem 8

seed 20260925

The number of household-insurance claims lodged per day with a large insurer follows a Poisson distribution with mean $\lambda = 108$.

The claims department wants an interval, centred at the mean, that is guaranteed by Tchebysheff's theorem to contain the daily count on at least $24/25$ of all days.

Give every number to at least four significant figures.

(a) The standard deviation of the daily count is $\sigma =$ _____

(b) The interval runs from _____ to _____

(c) Over a review period of 264 working days, the expected number of days on which the count falls inside this interval is at least _____

Problem 9

seed 20260925

A loan officer reviews 4 credit applications from unrelated applicants this morning. Each application, independently of the others, is approved with probability 0.555. Let Y be the number approved.

Give every number to at least four significant figures.

(a) $\mu =$ _____ and $\sigma =$ _____

(b) Using the binomial probability function, the exact probability $P(|Y - \mu| < \sigma) =$ _____

(c) The exact probability $P(|Y - \mu| < 2\sigma) =$ _____

(d) Tchebysheff's theorem guarantees $P(|Y - \mu| < 2\sigma) \geq 0.75$. By how much does the exact probability in (c) exceed this bound? _____

Problem 10

seed 20260925

The container terminal at the Port of Baku (Alat) handles Y containers per day, with mean $\mu = 380$ and standard deviation $\sigma = 38$. Nothing else is assumed about the distribution.

The terminal's staffing plan works well when $285 < Y < 532$.

Give every probability to at least four significant figures.

(a) The staffing interval is not centred at μ. The largest interval of the form $(\mu - k\sigma,\ \mu + k\sigma)$ that fits inside it has $k =$ _____

(b) Hence Tchebysheff's theorem gives the lower bound $P(285 < Y < 532) \geq$ _____

(c) An upper bound for $P(Y \geq 532)$ from Theorem 3.14 is _____

(d) An upper bound for $P(Y \leq 285)$ from Theorem 3.14 is _____

Problem 11

seed 20260925

A card-payment processor in Baku declines, on average, $\mu = 800$ transactions a day. Its service agreement with a partner bank asks that the daily number Y of declines lie strictly between 655 and 945 (that is, within ± 145 of the mean) with probability at least 0.90.

The distribution of Y is unknown, so the processor wants the agreement to be guaranteed by Tchebysheff's theorem alone.

Give every number to at least four significant figures.

(a) The largest standard deviation σ for which Tchebysheff's theorem guarantees $P(|Y-\mu|<145)\geq 0.90$ is

_____

(b) In fact the standard deviation is $\sigma = 73.95$. The lower bound that Tchebysheff's theorem gives for $P(|Y-\mu|<145)$ is _____

(c) With $\sigma = 73.95$, does Tchebysheff's theorem alone guarantee the agreement?

- ○ Yes
- ○ No

(d) With $\sigma = 73.95$, the narrowest band $\mu \pm w$ that Tchebysheff's theorem guarantees to hold with probability at least 0.90 has $w =$ _____

Problem 12

seed 20260925

A Baku freight forwarder insures 9 export shipments this month, each with a different client on a different route. Each shipment, independently of the others, produces an insurance claim with probability 0.27. Let Y be the number of claims.

Give every number to at least four significant figures.

(a) The probability-generating function of Y is $P(t) =$ _____

(Enter a function of t.)

(b) Using Theorem 3.13, $E(Y) = P^{(1)}(1) =$ _____

(c) $\mu_{[2]} = E[Y(Y-1)] =$ _____

(d) $V(Y) =$ _____

(e) Evaluate $P^{(1)}(0)$; it equals $P(Y=1)$. _____

(f) Each claim, independently, is settled without litigation with probability 0.80. The probability that no claim this month goes to court (including the case of no claims) is $P(0.80)$: _____

Problem 13

seed 20260925

A bank models the month Y in which a newly issued credit card first has a missed payment. Each month, independently, the cardholder misses a payment with probability $p = 0.066$, so Y is geometric: $p(y) = q^{y-1}p$, $y = 1, 2, \ldots$, with $q = 1-p$.

Example 3.26 derived its probability-generating function, $P(t) = \dfrac{pt}{1-qt}$, valid for $t < 1/q$.

Give every number to at least four significant figures.

(a) $P(t)$ is finite for t below _____

(b) $\mu_{[1]} = P^{(1)}(1) =$ _____

(c) $\mu_{[2]} = P^{(2)}(1) =$ _____

(d) $V(Y) =$ _____

(e) When the first miss happens the bank books a provision of 2500 AZN in that month. With a monthly discount factor $v = 0.999$, the present value of the provision is $2500\,v^{Y}$. Its expected present value, $2500\,E(v^{Y}) = 2500\,P(v)$, is _____ AZN

Problem 14

seed 20260925

An internal auditor at a bank branch pulls 8 of the branch's 48 mortgage files at random, without replacement. Unknown to the auditor, exactly 16 of the 48 files have a documentation error. Let Y be the number of flawed files among those pulled.

Give every number to at least four significant figures, as a decimal.

(a) $P(Y = 0) =$ _____

(b) $P(Y \geq 2) =$ _____

(c) $E(Y) =$ _____

(d) $V(Y) =$ _____

(e) Had the auditor put each file back before drawing the next (sampling with replacement), Y would be binomial. Its variance would then be _____

Problem 15

seed 20260925

A relationship manager at a Baku investment bank is placing a new corporate bond issue. She contacts institutional clients one at a time, and each contact, independently of the others, results in a placement with probability 0.48. She stops after the 2nd placement. Let Y be the number of the contact on which the 2nd placement is made.

Give every number to at least four significant figures.

(a) $P(Y = 6) =$ _____

(b) The probability that she needs at most 3 contacts, $P(Y \leq 3) =$ _____

(c) $E(Y) =$ _____

(d) $V(Y) =$ _____

Problem 16

seed 20260925

A credit-risk analyst at a Baku bank fits a model for an integer-valued count Y attached to the bank's retail loan book and reports only its moment-generating function:

$$m(t) = (0.74 + 0.26e^{t})^{28}$$

Give every number to at least four significant figures.

(a) Which distribution does Y have?

- ○ Binomial
- ○ Geometric
- ○ Negative binomial
- ○ Hypergeometric
- ○ Poisson

(b) $E(Y) =$ _____

(c) $V(Y) =$ _____

(d) $P(Y = 2) =$ _____

Problem 17

seed 20260925

A market-making desk on the Baku Stock Exchange records Y, the number of order legs it executes in a one-minute window. The risk system does not report the probability function of Y directly, only its probability-generating function

$$P(t) = e^{1.60(t^{2}-1)}.$$

Give every number to at least four significant figures.

(a) $P(Y = 4) =$ _____

(b) $P(Y = 6) =$ _____

(c) $E(Y) =$ _____

(d) $\mu_{[2]} = E[Y(Y-1)] =$ _____

(e) $V(Y) =$ _____

(f) Each leg, independently, clears without manual intervention with probability 0.90. The probability that every leg executed in the window clears automatically (including the case of no legs) is _____

Problem 18

seed 20260925

A hedged arbitrage book at a Baku bank has a daily profit-and-loss Y (thousand AZN) with only three possible values:

y	-25	0	25
$p(y)$	0.095	0.810	0.095

Take $k = 2.5$ throughout. Give every number to at least four significant figures.

(a) $E(Y) = 0$ by symmetry. The standard deviation is $\sigma =$ _____

(b) Using the distribution itself, the exact probability $P(|Y - \mu| \geq 2.5\sigma) =$ _____

(c) The upper bound Tchebysheff's theorem gives for this probability is _____

(d) Keep the values $-25, 0, 25$ and the symmetry, but change the probability of each large move from 0.095 to a new value a. The value of a for which the exact probability $P(|Y - \mu| \geq 2.5\sigma)$ equals the Tchebysheff bound is $a =$ _____

Problem 19

seed 20260925

A card issuer's fraud team monitors a merchant category in which 1400 transactions were made yesterday. Each transaction, independently of the others, is fraudulent with probability 0.0015. Let Y be the number of fraudulent transactions.

Give every number to at least four significant figures, as a decimal.

(a) The exact probability of at most two frauds, $P(Y \leq 2) =$ _____

(b) The Poisson approximation to the same probability is _____

(c) The standard deviation of Y (exact, binomial) is $\sigma =$ _____

(d) An alarm is raised if $Y \geq 9$. Without using the shape of the distribution, Tchebysheff's theorem bounds the probability of an alarm by _____

Problem 20

seed 20260925

A logistics company in Sumgayit leases a fleet of delivery vans. The number Y of breakdowns in a month is Poisson with mean $\lambda = 3.6$. The maintenance contract costs, for the month,

- a fixed fee of 6000 AZN,
- 500 AZN for every breakdown, and
- a surcharge of 400 AZN for every **pair** of breakdowns in the same month, because overlapping repairs need an extra crew.

So the monthly cost is $C = 6000 + 500Y + 400 \cdot \dfrac{Y(Y-1)}{2}$.

Give every number to at least four significant figures.

(a) Using the probability-generating function of the Poisson distribution, $\mu_{[2]} = E[Y(Y-1)] =$

_____

(b) $E(C) =$ _____ AZN

(c) The finance team wants a monthly budget that is exceeded with probability at most $4/25$, guaranteed by Tchebysheff's theorem. First find the upper end u of the interval $\lambda \pm k\sigma$ with $1 - 1/k^2 = 21/25$: $u =$

_____

(d) The cost at $Y = u$, the budget level, is _____ AZN