Week 9, Problem Set 1

Wackerly 4.1-4.2 - Continuous Random Variables: Distribution and Density Functions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku microfinance lender tracks Y, the number of monthly instalments a new borrower misses during the first quarter of a loan, so Y takes the values 0, 1, 2 or 3. Its records give the probability function

$$p(0) = 0.74, \qquad p(1) = 0.12, \qquad p(2) = 0.10, \qquad p(3) = 0.04.$$

Let $F(y) = P(Y \le y)$ be the distribution function of Y (Definition 4.1), defined for every real y.

(a) $F(-0.5) =$ _____

(b) $F(0.8) =$ _____

(c) $F(2.4) =$ _____

(d) $P(Y < 2) =$ _____

(e) The distribution function jumps at $y = 2$. The size of that jump, $F(2) - \lim_{y \uparrow 2} F(y)$, is _____

Problem 2

seed 20260925

After a trade on the Baku Stock Exchange is matched, the central depository needs some time to transfer the securities and the cash. Let Y be that settlement time in hours. The depository's model gives Y the distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ \left(\dfrac{y}{7}\right)^2, & 0 \le y \le 7, \\ 1, & y > 7. \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(Y \le 2.9) =$ _____

(b) $P(2.9 < Y \le 4.2) =$ _____

(c) The probability that settlement takes longer than 4.2 hours: _____

(d) $P(Y = 2.9) =$ _____

Problem 3

seed 20260925

An insurer's motor-claims department guarantees a decision within 19 days. The time Y (in days) from the filing of a claim to the decision is modelled by the density

$$f(y) = \begin{cases} k\,y^{1}, & 0 \le y \le 19, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures (a decimal or a fraction such as 3/64 is fine).

(a) The value of k that makes f a density: $k =$ _____

(b) The probability that a claim is decided within 8.55 days: _____

Problem 4

seed 20260925

A bank studies Y, the fraction of the credit limit that a cardholder has drawn at the monthly statement date ($Y = 0$: nothing drawn, $Y = 1$: limit fully used). Across its portfolio Y has density

$$f(y) = \begin{cases} 12\,y^{2}(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(0.13 \le Y \le 0.91) =$ _____

(b) $P(0.13 < Y < 0.91) =$ _____

(c) Among cardholders who have drawn at most 0.91 of their limit, the probability that a cardholder has drawn at most 0.13 of it: $P(Y \le 0.13 \mid Y \le 0.91) =$ _____

Problem 5

seed 20260925

A mobile operator charges prepaid subscribers for data used beyond their bundle. Let Y be one subscriber's overage charge (AZN) for a month. The operator's model gives Y the distribution function below.

$$F(y) = \begin{cases} 0, & y < 0, \\ 0.66, & 0 \le y < 5, \\ 0.82, & 5 \le y < 10, \\ 1, & y \ge 10. \end{cases}$$

(a) Using Definition 4.2 and what a discrete distribution function looks like, Y is

- ○ Discrete
- ○ Continuous
- ○ Neither discrete nor continuous

(b) $P(Y = 0) =$ _____

(c) $P(Y \le 7) =$ _____

Give probabilities to at least four significant figures.

Problem 6

seed 20260925

A bank's customer hotline guarantees that every caller reaches an agent within 19 minutes. The waiting time Y (minutes) of a caller has distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - \left(1 - \dfrac{y}{19}\right)^2, & 0 \le y \le 19, \\ 1, & y > 19. \end{cases}$$

(a) The density of Y for $0 < y < 19$: $f(y) =$ _____ (enter a formula in y, e.g. 3*(1-y/10)^2/10)

(b) The probability that a caller waits more than 3.25 minutes: _____

(c) The median waiting time $\phi_{.5}$: _____ minutes

Give numerical answers to at least four significant figures.

Problem 7

seed 20260925

Interbank payments in Azerbaijan's real-time settlement system are queued and cleared in batches. The clearing time Y (minutes) of a batch never exceeds 7.5 minutes, and has density

$$f(y) = \begin{cases} c\,(y + 1), & 0 \le y \le 7.5, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ _____

(b) The distribution function for $0 \le y \le 7.5$: $F(y) =$ _____ (a formula in y; you may use your value of c as a decimal or fraction)

(c) Use F to find $P(1.5 \le Y \le 6.75) =$ _____

Problem 8

seed 20260925

A Sumqayit steel distributor sells to builders on trade credit: invoices must be settled within 36 days. Most customers pay early to keep their credit line, so the payment time Y (days after invoicing) is modelled by

$$f(y) = \begin{cases} c\,(36 - y), & 0 \le y \le 36, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ _____

(b) For $0 \le y \le 36$: $F(y) =$ _____ (a formula in y)

(c) The probability that an invoice is paid between day 15 and day 24: _____

(d) The probability that payment takes more than 24 days: _____

Problem 9

seed 20260925

A Baku mobile operator sells a monthly data bundle. Let Y be the fraction of the bundle a subscriber uses during the month. The operator's analysts fit the density

$$f(y) = \begin{cases} c\,y^2 + 0.5\,y, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ _____

(b) The probability that a subscriber uses less than 0.65 of the bundle: _____

(c) Given that a subscriber has used at least 0.15 of the bundle, the probability that she uses at least 0.80 of it: _____

Problem 10

seed 20260925

An ATM in a Baku shopping centre is loaded each morning with 1.8 (tens of thousands of AZN) in cash. Let Y be the amount withdrawn in a day, in tens of thousands of AZN. Demand rises steadily up to 1 and then levels off, and the bank models it with the density

$$f(y) = \begin{cases} k\,y, & 0 \le y \le 1, \\ k, & 1 < y \le 1.8, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $k =$ _____

(b) $P(0 \le Y \le 0.9) =$ _____

(c) $P(0.6 \le Y \le 1.4) =$ _____

(d) The probability that more than 1.4 (tens of thousands of AZN) is withdrawn: _____

Problem 11

seed 20260925

The monthly turnover Y (thousands of USD) of a currency-exchange office in Baku's Fountain Square is modelled by the triangular density

$$f(y) = \begin{cases} \dfrac{y}{784}, & 0 < y < 28, \\ \dfrac{56 - y}{784}, & 28 \le y < 56, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $F(14) =$ _____

(b) The probability that turnover is between 14 and 35 thousand USD: _____

(c) Given that turnover exceeds 28 thousand USD, the probability that it exceeds 42 thousand USD: _____

Problem 12

seed 20260925

When a small-business loan defaults, the bank eventually recovers a fraction Y of the outstanding balance through collateral sales and negotiation. A Baku bank's workout unit models the recovery rate by

$$f(y) = \begin{cases} 6\,y(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures. (The quantile equations are cubics; solve them numerically, e.g. by trial and error on a calculator.)

(a) The probability that the bank recovers at most 0.84 of the balance: _____

(b) The 0.13-quantile of Y, $\phi_{0.13}$: _____

(c) The 0.87-quantile, $\phi_{0.87}$: _____

Problem 13

seed 20260925

A broker splits a large block order for a bank's shares into small child orders to avoid moving the price on the Baku Stock Exchange. The time Y (hours) until the whole block is filled has distribution function

$$F(y) = \begin{cases} 0, & y \le 0, \\ \dfrac{1\,y}{25}, & 0 < y < 1, \\ \dfrac{y^2}{25}, & 1 \le y < 5, \\ 1, & y \ge 5. \end{cases}$$

Give numerical answers to at least four significant figures.

(a) The density for $0 < y < 1$: $f(y) =$ _____

(b) The density for $1 < y < 5$: $f(y) =$ _____ (a formula in y)

(c) $P(0.4 \le Y \le 4.2) =$ _____

(d) $P(Y \ge 0.4 \mid Y \le 4.2) =$ _____

Problem 14

seed 20260925

A mobile operator sells prepaid data bundles priced at 3, 5, 8, 12 and 20 AZN. Let Y be the price of the bundle bought by a randomly chosen top-up. From last month's sales, the distribution function of Y is

$$F(y) = \begin{cases} 0, & y < 3, \\ 0.07, & 3 \le y < 5, \\ 0.36, & 5 \le y < 8, \\ 0.57, & 8 \le y < 12, \\ 0.86, & 12 \le y < 20, \\ 1, & y \ge 20. \end{cases}$$

(a) $P(Y = 8) =$ _____

(b) $P(5 < Y < 12) =$ _____

(c) $P(5 \le Y < 12) =$ _____

(d) The median $\phi_{.5}$ of Y (Definition 4.4): _____ AZN

(e) The 0.9-quantile $\phi_{.9}$: _____ AZN

Problem 15

seed 20260925

A bank installs card-payment (POS) terminals in shops across Ganja. The time Y (years) until a terminal fails has distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2/4.84}, & y \ge 0. \end{cases}$$

Give numerical answers to at least four significant figures.

(a) The 0.50-quantile $\phi_{0.50}$ (the age by which 0.50 of terminals have failed): _____ years

(b) The density for $y > 0$: $f(y) =$ _____ (a formula in y; type e^u as e^(u))

(c) The probability that a terminal is still working after 1.5 years: _____

(d) The bank replaces every terminal that has failed within 3 years under warranty. Among warranty replacements, the fraction that failed after more than 1.5 years, $P(Y > 1.5 \mid Y \le 3)$: _____

Problem 16

seed 20260925

A bank records every operational-loss event (fraud, system outage, processing error) of at least 5 thousand AZN. The size Y (thousand AZN) of a recorded loss is modelled by

$$f(y) = \begin{cases} \dfrac{5}{y^2}, & y \ge 5, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) For $y \ge 5$: $F(y) =$ _____ (a formula in y)

(b) The probability that a recorded loss exceeds 15 thousand AZN: _____

(c) $P(Y > 23 \mid Y > 15) =$ _____

(d) The median loss $\phi_{.5}$: _____ thousand AZN

Problem 17

seed 20260925

The Baku Stock Exchange limits the daily price move of a listed manat corporate bond to $\pm 1\%$. Let Y be the bond's price change on a day, in percent. An analyst models it by

$$f(y) = \begin{cases} 0.20, & -1 < y \le 0, \\ 0.20 + c\,y, & 0 < y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ _____

(b) $F(0) =$ _____

(c) For $0 < y \le 1$: $F(y) =$ _____ (a formula in y)

(d) $P(0 \le Y \le 0.8) =$ _____

(e) Given that the bond gains more than 0.15% on a day, the probability that it gains more than 0.9%: $P(Y > 0.9 \mid Y > 0.15) =$ _____

Problem 18

seed 20260925

When a consumer loan at a Baku bank defaults, the fraction of the exposure the bank finally loses is the loss given default, L. Some defaulted borrowers cure and repay in full ($L = 0$); for some the bank recovers nothing ($L = 1$). The bank's model gives L the distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 0.19 + 0.48\,y^{2}, & 0 \le y < 1, \\ 1, & y \ge 1. \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $P(L = 0) =$ _____

(b) $P(L = 1) =$ _____

(c) $P(L \le 0.20) =$ _____

(d) $P(0.20 < L \le 0.85) =$ _____

(e) The median $\phi_{.5}$ of L (Definition 4.4): _____

(f) The 0.9-quantile $\phi_{.9}$: _____

Problem 19

seed 20260925

A corporate client of a Baku bank has a revolving credit line. Let Y be the fraction of the line in use at a month-end. The bank's model is the triangular density with peak at $m = 0.37$:

$$f(y) = \begin{cases} \dfrac{2y}{0.37}, & 0 \le y \le 0.37, \\ \dfrac{2(1-y)}{0.63}, & 0.37 < y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) For $0.37 < y \le 1$: $F(y) =$ _____ (a formula in y)

(b) The median utilisation $\phi_{.5}$: _____

(c) $\phi_{0.19}$ (only 0.19 of month-ends have lower utilisation): _____

(d) $\phi_{0.89}$ (the bank's "high-usage" alert level): _____

Problem 20

seed 20260925

A Baku bank sends a customer's USD transfer abroad through a chain of correspondent banks. The time Y (hours) until the beneficiary is credited is never more than 8 hours, and the bank models it by

$$f(y) = \begin{cases} k\,y^2(8-y), & 0 \le y \le 8, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $k =$ _____

(b) For $0 \le y \le 8$: $F(y) =$ _____ (a formula in y)

(c) A customer calls after 3.2 hours: the money has not arrived. The probability it is still not credited after 5.2 hours: $P(Y > 5.2 \mid Y > 3.2) =$ _____

(d) The bank wants to quote a time by which 0.80 of transfers are credited, the quantile $\phi_{0.80}$: _____ hours