Week 9, Problem Set 1 - worked solutions

Wackerly 4.1-4.2 - Continuous Random Variables: Distribution and Density Functions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku microfinance lender tracks Y, the number of monthly instalments a new borrower misses during the first quarter of a loan, so Y takes the values 0, 1, 2 or 3. Its records give the probability function

$$p(0) = 0.74, \qquad p(1) = 0.12, \qquad p(2) = 0.10, \qquad p(3) = 0.04.$$

Let $F(y) = P(Y \le y)$ be the distribution function of Y (Definition 4.1), defined for every real y.

(a) $F(-0.5) =$ ______________

(b) $F(0.8) =$ ______________

(c) $F(2.4) =$ ______________

(d) $P(Y < 2) =$ ______________

(e) The distribution function jumps at $y = 2$. The size of that jump, $F(2) - \lim_{y \uparrow 2} F(y)$, is ______________

By Definition 4.1, $F(y)$ adds up $p(k)$ over the possible values $k \le y$. Because Y only takes the values 0, 1, 2, 3, F is a step function:

$$F(y) = \begin{cases} 0, & y < 0, \\ 0.74, & 0 \le y < 1, \\ 0.86, & 1 \le y < 2, \\ 0.96, & 2 \le y < 3, \\ 1, & y \ge 3. \end{cases}$$

(a) No possible value of Y is at most -0.5, so $F(-0.5) = 0$.

(b) The only possible value at most 0.8 is 0, so $F(0.8) = p(0) = 0.74$. The function is flat between 0 and 1: nothing happens there.

(c) $F(2.4) = p(0) + p(1) + p(2) = 0.74 + 0.12 + 0.10 = 0.96$.

(d) $P(Y < 2) = p(0) + p(1) = 0.86$. Note that this is **not** $F(2)$: $F(2) = P(Y \le 2) = 0.96$ also includes $p(2)$.

(e) Just to the left of 2, $F(y) = 0.86$; at 2 it equals 0.96. The jump is $0.96 - 0.86 = 0.10 = P(Y = 2)$. For a discrete variable the distribution function rises only by jumps, and each jump is the probability of the point where it happens. A continuous variable (Definition 4.2) has no jumps, which is why $P(Y = y) = 0$ for every y in that case.

Problem 2

seed 20260925

After a trade on the Baku Stock Exchange is matched, the central depository needs some time to transfer the securities and the cash. Let Y be that settlement time in hours. The depository's model gives Y the distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ \left(\frac{y}{7}\right)^2, & 0 \le y \le 7, \\ 1, & y > 7. \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(Y \le 2.9) =$ ____________

(b) $P(2.9 < Y \le 4.2) =$ ____________

(c) The probability that settlement takes longer than 4.2 hours: ____________

(d) $P(Y = 2.9) =$ ____________

F is continuous everywhere (at $y = 0$ both pieces give 0 and at $y = 7$ both give 1), so Y is a continuous random variable by Definition 4.2.

(a) By Definition 4.1, $P(Y \le 2.9) = F(2.9) = (2.9/7)^2 = 0.1716$.

(b) $P(a < Y \le b) = F(b) - F(a)$, so

$$P(2.9 < Y \le 4.2) = \left(\frac{4.2}{7}\right)^2 - \left(\frac{2.9}{7}\right)^2 = 0.3600 - 0.1716 = 0.1884.$$

(c) $P(Y > 4.2) = 1 - F(4.2) = 1 - 0.3600 = 0.6400$.

(d) A continuous distribution function has no jumps, so $P(Y = 2.9) = F(2.9) - \lim_{y \uparrow 2.9} F(y) = 0$. This is also why the answer to (b) would not change if either $<$ were replaced by $\le$ (the remark after Theorem 4.3).

Problem 3

seed 20260925

An insurer's motor-claims department guarantees a decision within 19 days. The time Y (in days) from the filing of a claim to the decision is modelled by the density

$$f(y) = \begin{cases} k\,y^1, & 0 \le y \le 19, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures (a decimal or a fraction such as 3/64 is fine).

(a) The value of k that makes f a density: $k =$ ____________

(b) The probability that a claim is decided within 8.55 days: ____________

(a) For $k > 0$, $f(y) \ge 0$, so by Theorem 4.2 the only requirement is that the total area be 1:

$$1 = \int_{-\infty}^{\infty} f(y)\,dy = \int_0^{19} k\,y^1\,dy = k\,\frac{y^2}{2}\bigg|_0^{19} = k\,\frac{19^2}{2},$$

so $k = 2/19^2 = 2/361 = 0.00554017$.

(b) By Theorem 4.3,

$$P(0 \le Y \le 8.55) = \int_0^{8.55} \frac{2}{19^2}\, y^1\, dy = \left(\frac{8.55}{19}\right)^2 = 0.45^2 = 0.2025.$$

Because the density rises with y, most of the probability sits near the 19-day limit: a claim is decided within the first 0.45 of the allowed time with probability only 0.2025.

Problem 4

seed 20260925

A bank studies Y, the fraction of the credit limit that a cardholder has drawn at the monthly statement date ($Y = 0$: nothing drawn, $Y = 1$: limit fully used). Across its portfolio Y has density

$$f(y) = \begin{cases} 12\, y^2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(0.13 \le Y \le 0.91) =$ ______________

(b) $P(0.13 < Y < 0.91) =$ ______________

(c) Among cardholders who have drawn at most 0.91 of their limit, the probability that a cardholder has drawn at most 0.13 of it: $P(Y \le 0.13 \mid Y \le 0.91) =$ ______________

First find the distribution function on $[0, 1]$:

$$F(y) = \int_0^y 12\, t^2(1-t)\, dt = 12\left(\frac{y^3}{3} - \frac{y^4}{4}\right) = 4y^3 - 3y^4, \qquad 0 \le y \le 1.$$

So $F(0.13) = 0.007931$ and $F(0.91) = 0.957035$.

(a) By Theorem 4.3, $P(0.13 \le Y \le 0.91) = \int_{0.13}^{0.91} f(y)\, dy = F(0.91) - F(0.13) = 0.9491$.

(b) The same, 0.9491. Y is continuous, so $P(Y = 0.13) = P(Y = 0.91) = 0$ and whether the endpoints are included makes no difference. (For a discrete variable it can: compare Exercise 4.7.)

(c) By the definition of conditional probability, and since $\{Y \le 0.13\} \subset \{Y \le 0.91\}$,

$$P(Y \le 0.13 \mid Y \le 0.91) = \frac{P(Y \le 0.13)}{P(Y \le 0.91)} = \frac{F(0.13)}{F(0.91)} = \frac{0.007931}{0.957035} = 0.0083.$$

Problem 5

seed 20260925

A mobile operator charges prepaid subscribers for data used beyond their bundle. Let Y be one subscriber's overage charge (AZN) for a month. The operator's model gives Y the distribution function below.

$$F(y) = \begin{cases} 0, & y < 0, \\ 0.66, & 0 \le y < 5, \\ 0.82, & 5 \le y < 10, \\ 1, & y \ge 10. \end{cases}$$

(a) Using Definition 4.2 and what a discrete distribution function looks like, Y is

- ◯ Discrete
- ◯ Continuous
- ◯ Neither discrete nor continuous

(b) $P(Y = 0) =$ ______________

(c) $P(Y \le 7) =$ ______________

Give probabilities to at least four significant figures.

(a) F is a step function: it is flat except for jumps at 0, 5 and 10. That is the distribution function of a discrete variable (Example 4.1): the operator charges overage in whole 5-AZN blocks, so Y takes only the values 0, 5 and 10. It is not continuous in the sense of Definition 4.2 because F has jumps.

(b) $P(Y = 0)$ is the size of the jump at 0: $F(0) - \lim_{y \uparrow 0} F(y) = 0.66 - 0 = 0.66$.

(c) $P(Y \le 7) = F(7) = 0.82$, read off the flat piece $5 \le y < 10$.

Problem 6

seed 20260925

A bank's customer hotline guarantees that every caller reaches an agent within 19 minutes. The waiting time Y (minutes) of a caller has distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - \left(1 - \frac{y}{19}\right)^2, & 0 \le y \le 19, \\ 1, & y > 19. \end{cases}$$

(a) The density of Y for $0 < y < 19$: $f(y) =$ ______________ (enter a formula in y, e.g. 3*(1-y/10)^2/10)

(b) The probability that a caller waits more than 3.25 minutes: ______________

(c) The median waiting time $\phi_{.5}$: ______________ minutes

Give numerical answers to at least four significant figures.

(a) By Definition 4.3, $f(y) = F'(y)$ wherever the derivative exists. By the chain rule, for $0 < y < 19$,

$$f(y) = \frac{d}{dy}\left[1 - \left(1 - \frac{y}{19}\right)^2\right] = 2\left(1 - \frac{y}{19}\right)^1 \cdot \frac{1}{19} = \frac{2}{19}\left(1 - \frac{y}{19}\right)^1,$$

and $f(y) = 0$ outside $[0, 19]$. The density is largest at $y = 0$: short waits are the most common.

(b) $P(Y > 3.25) = 1 - F(3.25) = (1 - 3.25/19)^2 = 0.8289^2 = 0.6872$.

(c) By Definition 4.4, for a continuous Y the median solves $F(\phi_{.5}) = .5$:

$$1-\left(1-\frac{\phi_{.5}}{19}\right)^2 = 0.5 \implies 1-\frac{\phi_{.5}}{19} = 0.5^{1/2} = 0.707107 \implies \phi_{.5} = 19(1-0.707107) = 5.5650.$$

Problem 7

seed 20260925

Interbank payments in Azerbaijan's real-time settlement system are queued and cleared in batches. The clearing time Y (minutes) of a batch never exceeds 7.5 minutes, and has density

$$f(y) = \begin{cases} c\,(y+1), & 0 \le y \le 7.5, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ ______________

(b) The distribution function for $0 \le y \le 7.5$: $F(y) =$ ______________ (a formula in y; you may use your value of c as a decimal or fraction)

(c) Use F to find $P(1.5 \le Y \le 6.75) =$ ______________

(a) For $c > 0$ the density is non-negative on $[0, 7.5]$, so Theorem 4.2 only requires total area 1:

$$1 = \int_0^{7.5} c\,(y+1)\,dy = c\left[\frac{y^2}{2} + 1y\right]_0^{7.5} = c\left(\frac{7.5^2}{2} + 1\cdot 7.5\right) = 35.625\,c,$$

so $c = 1/35.625 = 0.028070$.

(b) For $0 \le y \le 7.5$,

$$F(y) = \int_{-\infty}^{y} f(t)\,dt = \int_0^y c\,(t+1)\,dt = \frac{1}{35.625}\left(\frac{y^2}{2} + 1y\right),$$

with $F(y) = 0$ for $y < 0$ and $F(y) = 1$ for $y > 7.5$. Check: $F(7.5) = 35.625/35.625 = 1$.

(c) By Theorem 4.3 (and $P(Y = 1.5) = 0$),

$$P(1.5 \le Y \le 6.75) = F(6.75) - F(1.5) = 0.8289 - 0.0737 = 0.7553.$$

Problem 8

seed 20260925

A Sumqayit steel distributor sells to builders on trade credit: invoices must be settled within 36 days. Most customers pay early to keep their credit line, so the payment time Y (days after invoicing) is modelled by

$$f(y) = \begin{cases} c\,(36-y), & 0 \le y \le 36, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ ______________

(b) For $0 \le y \le 36$: $F(y) =$ ______________ (a formula in y)

(c) The probability that an invoice is paid between day 15 and day 24: _______________

(d) The probability that payment takes more than 24 days: _______________

(a) The graph of f is a triangle with base 36 and height $c \cdot 36$. Theorem 4.2 needs the area to be 1:

$$\int_0^{36} c(36-y)\,dy = c\left[36y - \frac{y^2}{2}\right]_0^{36} = c\,\frac{36^2}{2} = 1 \implies c = \frac{2}{1296} = 0.001543.$$

(b) For $0 \le y \le 36$,

$$F(y) = \int_0^y \frac{2}{1296}(36-t)\,dt = \frac{2}{1296}\left(36y - \frac{y^2}{2}\right) = \frac{2 \cdot 36\,y - y^2}{1296},$$

with $F(y) = 0$ for $y < 0$ and $F(y) = 1$ for $y > 36$.

(c) $P(15 \le Y \le 24) = F(24) - F(15) = 0.8889 - 0.6597 = 0.2292$ (Theorem 4.3).

(d) $P(Y > 24) = 1 - F(24)$. Or by geometry: the region under f to the right of 24 is a triangle with base $36 - 24 = 12$ and height $f(24) = 0.018519$, so its area is

$$\tfrac{1}{2} \cdot 12 \cdot 0.018519 = \left(\frac{12}{36}\right)^2 = 0.1111.$$

Problem 9

seed 20260925

A Baku mobile operator sells a monthly data bundle. Let Y be the fraction of the bundle a subscriber uses during the month. The operator's analysts fit the density

$$f(y) = \begin{cases} c\,y^2 + 0.5\,y, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ _______________

(b) The probability that a subscriber uses less than 0.65 of the bundle: _______________

(c) Given that a subscriber has used at least 0.15 of the bundle, the probability that she uses at least 0.80 of it:

____

(a) By Theorem 4.2,

$$1 = \int_0^1 (c\,y^2 + 0.5\,y)\,dy = \frac{c}{3} + \frac{0.5}{2} \implies c = 3\left(1 - \frac{0.5}{2}\right) = 2.25.$$

With $c > 0$ the density is non-negative on $[0, 1]$, so it is valid.

(b) For $0 \le y \le 1$,

$$F(y) = \int_0^y (c\,t^2 + 0.5\,t)\,dt = \frac{c}{3}y^3 + \frac{0.5}{2}y^2 = 0.7500\,y^3 + 0.25\,y^2.$$

Because Y is continuous, $P(Y < 0.65) = P(Y \leq 0.65) = F(0.65) = 0.3116$.

(c) The event $\{Y \geq 0.80\}$ is contained in $\{Y \geq 0.15\}$, so

$$P(Y \geq 0.80 \mid Y \geq 0.15) = \frac{1 - F(0.80)}{1 - F(0.15)} = \frac{1 - 0.544000}{1 - 0.008156} = \frac{0.456000}{0.991844} = 0.4597.$$

Problem 10

seed 20260925

An ATM in a Baku shopping centre is loaded each morning with 1.8 (tens of thousands of AZN) in cash. Let Y be the amount withdrawn in a day, in tens of thousands of AZN. Demand rises steadily up to 1 and then levels off, and the bank models it with the density

$$f(y) = \begin{cases} k\,y, & 0 \leq y \leq 1, \\ k, & 1 < y \leq 1.8, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $k =$ ______________

(b) $P(0 \leq Y \leq 0.9) =$ ______________

(c) $P(0.6 \leq Y \leq 1.4) =$ ______________

(d) The probability that more than 1.4 (tens of thousands of AZN) is withdrawn: ______________

(a) The area under f is a triangle on $[0, 1]$ plus a rectangle on $(1, 1.8]$. By Theorem 4.2,

$$1 = \int_0^1 k\,y\,dy + \int_1^{1.8} k\,dy = \frac{k}{2} + k(1.8 - 1) = k(1.8 - 0.5) = 1.3\,k,$$

so $k = 1/1.3 = 0.769231$.

(b) By Theorem 4.3, $P(0 \leq Y \leq 0.9) = \int_0^{0.9} k\,y\,dy = k\,\frac{0.9^2}{2} = 0.3115$.

(c) The interval crosses the kink at 1, so split the integral there:

$$\int_{0.6}^1 k\,y\,dy + \int_1^{1.4} k\,dy = \frac{k}{2}(1 - 0.6^2) + k(1.4 - 1) = \frac{k}{2}(0.64) + k(0.4) = 0.2462 + 0.3077 = 0.5538.$$

(d) $P(Y > 1.4) = \int_{1.4}^{1.8} k\,dy = k(1.8 - 1.4) = k(0.4) = 0.3077$.

Problem 11

seed 20260925

The monthly turnover Y (thousands of USD) of a currency-exchange office in Baku's Fountain Square is modelled by the triangular density

$$f(y) = \begin{cases} \dfrac{y}{784}, & 0 < y < 28, \\ \dfrac{56 - y}{784}, & 28 \leq y < 56, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $F(14) =$ _______________

(b) The probability that turnover is between 14 and 35 thousand USD: _______________

(c) Given that turnover exceeds 28 thousand USD, the probability that it exceeds 42 thousand USD:

____

Integrate the density piece by piece (note $784 = 28^2$). For $0 \le y < 28$:

$$F(y) = \int_0^y \frac{t}{784}\,dt = \frac{y^2}{2 \cdot 784}.$$

So $F(28) = 1/2$, as symmetry demands. For $28 \le y < 56$:

$$F(y) = \frac{1}{2} + \int_{28}^y \frac{56-t}{784}\,dt = 1 - \frac{(56-y)^2}{2 \cdot 784},$$

and $F(y) = 0$ for $y < 0$, $F(y) = 1$ for $y \ge 56$.

(a) $F(14) = 14^2/(2 \cdot 784) = 0.1250$.

(b) $y_2 = 35$ is on the right-hand piece: $F(35) = 1 - 21^2/(2 \cdot 784) = 0.7188$, so by Theorem 4.3

$$P(14 \le Y \le 35) = F(35) - F(14) = 0.7188 - 0.1250 = 0.5938.$$

(c) $P(Y > 42) = 1 - F(42) = 14^2/(2 \cdot 784) = 0.125000$ and $P(Y > 28) = 1/2$, so

$$P(Y > 42 \mid Y > 28) = \frac{0.125000}{0.5} = \left(\frac{14}{28}\right)^2 = 0.2500.$$

Problem 12

seed 20260925

When a small-business loan defaults, the bank eventually recovers a fraction Y of the outstanding balance through collateral sales and negotiation. A Baku bank's workout unit models the recovery rate by

$$f(y) = \begin{cases} 6\,y(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures. (The quantile equations are cubics; solve them numerically, e.g. by trial and error on a calculator.)

(a) The probability that the bank recovers at most 0.84 of the balance: _______________

(b) The 0.13-quantile of Y, $\phi_{0.13}$: _______________

(c) The 0.87-quantile, $\phi_{0.87}$: _______________

For $0 \le y \le 1$,

$$F(y) = \int_0^y 6t(1-t)\,dt = 3y^2 - 2y^3.$$

(a) $P(Y \le 0.84) = F(0.84) = 3(0.84)^2 - 2(0.84)^3 = 0.9314$.

(b) Y is continuous, so by Definition 4.4 $\phi_{0.13}$ solves $F(\phi) = 0.13$:

$$3\phi^2 - 2\phi^3 = 0.13.$$

Since $0.13 < 0.5 = F(0.5)$, the root lies in $(0, 0.5)$, and F is increasing there, so there is exactly one. Bisection (or Newton's method) gives $\phi_{0.13} = 0.2259$; check: $3(0.2259)^2 - 2(0.2259)^3 = 0.1300$.

(c) The density is symmetric about 0.5: $f(1-y) = f(y)$. Hence $P(Y \ge 1-\phi) = P(Y \le \phi) = 0.13$, i.e. $F(1-\phi_{0.13}) = 0.87$. So

$$\phi_{0.87} = 1 - \phi_{0.13} = 1 - 0.2259 = 0.7741.$$

The bank can say that in 0.13 of defaults it recovers less than 0.2259 of the balance, and in the same proportion more than 0.7741.

Problem 13

seed 20260925

A broker splits a large block order for a bank's shares into small child orders to avoid moving the price on the Baku Stock Exchange. The time Y (hours) until the whole block is filled has distribution function

$$F(y) = \begin{cases} 0, & y \le 0, \\ \frac{1\,y}{25}, & 0 < y < 1, \\ \frac{y^2}{25}, & 1 \le y < 5, \\ 1, & y \ge 5. \end{cases}$$

Give numerical answers to at least four significant figures.

(a) The density for $0 < y < 1$: $f(y) =$ ______________

(b) The density for $1 < y < 5$: $f(y) =$ ______________ (a formula in y)

(c) $P(0.4 \le Y \le 4.2) =$ ______________

(d) $P(Y \ge 0.4 \mid Y \le 4.2) =$ ______________

First note that F is continuous: at $y = 1$ both middle pieces give $1^2/25$, and at $y = 5$ the last piece gives $25/25 = 1$. So Y is continuous and has a density.

(a), **(b)** By Definition 4.3, $f(y) = F'(y)$:

$$f(y) = \begin{cases} 1/25 = 0.040000, & 0 < y < 1, \\ 2y/25, & 1 < y < 5, \\ 0, & \text{elsewhere}, \end{cases}$$

undefined at the kinks $y = 0, 1, 5$ (which does not matter for any probability).

(c) 0.4 lies on the first piece and 4.2 on the second, so

$$P(0.4 \le Y \le 4.2) = F(4.2) - F(0.4) = \frac{4.2^2}{25} - \frac{1 \cdot 0.4}{25} = 0.7056 - 0.0160 = 0.6896.$$

(d)

$$P(Y \ge 0.4 \mid Y \le 4.2) = \frac{P(0.4 \le Y \le 4.2)}{P(Y \le 4.2)} = \frac{0.6896}{0.7056} = 0.9773.$$

Problem 14

seed 20260925

A mobile operator sells prepaid data bundles priced at 3, 5, 8, 12 and 20 AZN. Let Y be the price of the bundle bought by a randomly chosen top-up. From last month's sales, the distribution function of Y is

$$F(y) = \begin{cases} 0, & y < 3, \\ 0.07, & 3 \le y < 5, \\ 0.36, & 5 \le y < 8, \\ 0.57, & 8 \le y < 12, \\ 0.86, & 12 \le y < 20, \\ 1, & y \ge 20. \end{cases}$$

(a) $P(Y = 8) =$ ______________

(b) $P(5 < Y < 12) =$ ______________

(c) $P(5 \le Y < 12) =$ ______________

(d) The median $\phi_{.5}$ of Y (Definition 4.4): ______________ AZN

(e) The 0.9-quantile $\phi_{.9}$: ______________ AZN

F is a step function, so Y is discrete; its possible values are the points where F jumps, and $p(y)$ is the height of the jump there.

(a) $P(Y = 8) = F(8) - \lim_{y \uparrow 8} F(y) = 0.57 - 0.36 = 0.21$.

(b) The only possible value strictly between 5 and 12 is 8, so $P(5 < Y < 12) = p(8) = 0.21$. Equivalently $\lim_{y \uparrow 12} F(y) - F(5) = 0.57 - 0.36$.

(c) Now 5 is included too: $P(5 \le Y < 12) = p(5) + p(8) = 0.57 - 0.07 = 0.50$. For a discrete variable, whether an endpoint is included matters; for a continuous one it would not (compare Exercise 4.7).

(d) Definition 4.4: $\phi_{.5}$ is the **smallest** value with $F(\phi_{.5}) \ge .5$. Reading up the steps, the first value at which F reaches .5 is $y = 8$, where $F = 0.57$. So $\phi_{.5} = 8$ AZN. There is usually no y with $F(y) = .5$ exactly, which is why the definition uses $\ge$.

(e) Likewise the first value at which $F \ge .9$ is $y = 20$ (where $F = 1$), so $\phi_{.9} = 20$ AZN.

Problem 15

seed 20260925

A bank installs card-payment (POS) terminals in shops across Ganja. The time Y (years) until a terminal fails has distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2/4.84}, & y \geq 0. \end{cases}$$

Give numerical answers to at least four significant figures.

(a) The 0.50-quantile $\phi_{0.50}$ (the age by which 0.50 of terminals have failed): ______________ years

(b) The density for $y > 0$: $f(y) =$ ______________ (a formula in y; type e^u as e^(u))

(c) The probability that a terminal is still working after 1.5 years: ______________

(d) The bank replaces every terminal that has failed within 3 years under warranty. Among warranty replacements, the fraction that failed after more than 1.5 years, $P(Y > 1.5 \mid Y \leq 3)$: ______________

(a) Y is continuous, so ϕ_p solves $F(\phi_p) = p$ (Definition 4.4):

$$1 - e^{-\phi^2/4.84} = 0.50 \implies \phi^2 = -4.84 \ln(1 - 0.50) = 4.84 \times 0.693147 \implies \phi_{0.50} = 2.2\sqrt{0.693147} = 1.8316.$$

(b) By Definition 4.3, for $y > 0$,

$$f(y) = F'(y) = \frac{2y}{4.84} e^{-y^2/4.84},$$

and $f(y) = 0$ for $y < 0$.

(c) $P(Y > 1.5) = 1 - F(1.5) = e^{-1.5^2/4.84} = 0.6282.$

(d) $F(1.5) = 0.371787$ and $F(3) = 0.844250$, so

$$P(Y > 1.5 \mid Y \leq 3) = \frac{F(3) - F(1.5)}{F(3)} = \frac{0.844250 - 0.371787}{0.844250} = 0.5596.$$

Problem 16

seed 20260925

A bank records every operational-loss event (fraud, system outage, processing error) of at least 5 thousand AZN. The size Y (thousand AZN) of a recorded loss is modelled by

$$f(y) = \begin{cases} \dfrac{5}{y^2}, & y \geq 5, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) For $y \geq 5$: $F(y) =$ ______________ (a formula in y)

(b) The probability that a recorded loss exceeds 15 thousand AZN: ______________

(c) $P(Y > 23 \mid Y > 15) =$ ______________

(d) The median loss $\phi_{.5}$: ______________ thousand AZN

$f(y) \geq 0$ everywhere, and $\int_5^\infty 5/y^2\, dy = [-5/y]_5^\infty = 0 + 1 = 1$, so f satisfies Theorem 4.2.

(a) For $y \geq 5$,

$$F(y) = \int_5^y \frac{5}{t^2}\,dt = \left[-\frac{5}{t}\right]_5^y = 1 - \frac{5}{y},$$

and $F(y) = 0$ for $y < 5$.

(b) $P(Y > 15) = 1 - F(15) = 5/15 = 0.3333$.

(c) $\{Y > 23\} \subset \{Y > 15\}$, so

$$P(Y > 23 \mid Y > 15) = \frac{1 - F(23)}{1 - F(15)} = \frac{5/23}{5/15} = \frac{15}{23} = 0.6522.$$

The larger the loss already is, the likelier it is to grow a further fixed amount: this density has a heavy tail, which is why operational risk is dominated by a few large events.

(d) $F(\phi_{.5}) = .5$ gives $1 - 5/\phi_{.5} = .5$, so $\phi_{.5} = 2 \times 5 = 10$ thousand AZN.

Problem 17

seed 20260925

The Baku Stock Exchange limits the daily price move of a listed manat corporate bond to $\pm 1\%$. Let Y be the bond's price change on a day, in percent. An analyst models it by

$$f(y) = \begin{cases} 0.20, & -1 < y \leq 0, \\ 0.20 + c\,y, & 0 < y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $c =$ ______________

(b) $F(0) =$ ______________

(c) For $0 < y \leq 1$: $F(y) =$ ______________ (a formula in y)

(d) $P(0 \leq Y \leq 0.8) =$ ______________

(e) Given that the bond gains more than 0.15% on a day, the probability that it gains more than 0.9%:
$P(Y > 0.9 \mid Y > 0.15) =$ ______________

(a) Theorem 4.2 requires

$$1 = \int_{-1}^{0} 0.20\,dy + \int_0^1 (0.20 + c\,y)\,dy = 0.20 + 0.20 + \frac{c}{2} \implies c = 2(1 - 2 \times 0.20) = 1.20.$$

We also need $f(y) \geq 0$. On $(0, 1]$ the density is linear, so it is enough to check the ends: $f(0^+) = 0.20$ and $f(1) = 0.20 + 1.20 = 1.40$, both non-negative.

(b) $F(0) = \int_{-1}^{0} 0.20\,dy = 0.20$.

(c) For $0 < y \leq 1$,

$$F(y) = F(0) + \int_0^y (0.20 + c\,t)\,dt = 0.20 + 0.20\,y + \frac{c}{2}\,y^2 = 0.20 + 0.20\,y + 0.60\,y^2.$$

(For $-1 < y \le 0$, $F(y) = 0.20(y+1)$; $F = 0$ below -1 and $F = 1$ above 1, and indeed $F(1) = 2 \times 0.20 + 0.60 = 1$.)

(d) $P(0 \le Y \le 0.8) = F(0.8) - F(0) = 0.7440 - 0.20 = 0.5440$.

(e) $F(0.15) = 0.243500$ and $F(0.9) = 0.866000$, so

$$P(Y > 0.9 \mid Y > 0.15) = \frac{1 - F(0.9)}{1 - F(0.15)} = \frac{0.134000}{0.756500} = 0.1771.$$

Problem 18

seed 20260925

When a consumer loan at a Baku bank defaults, the fraction of the exposure the bank finally loses is the loss given default, L. Some defaulted borrowers cure and repay in full ($L = 0$); for some the bank recovers nothing ($L = 1$). The bank's model gives L the distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 0.19 + 0.48\,y^2, & 0 \le y < 1, \\ 1, & y \ge 1. \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $P(L = 0) =$ ______________

(b) $P(L = 1) =$ ______________

(c) $P(L \le 0.20) =$ ______________

(d) $P(0.20 < L \le 0.85) =$ ______________

(e) The median $\phi_{.5}$ of L (Definition 4.4): ______________

(f) The 0.9-quantile $\phi_{.9}$: ______________

This F satisfies Theorem 4.1 (it runs from 0 to 1 and never decreases), but it has jumps at 0 and at 1 and rises continuously in between: L is neither discrete nor continuous. The size of a jump is the probability of that point.

(a) $P(L = 0) = F(0) - \lim_{y \uparrow 0} F(y) = 0.19 - 0 = 0.19$.

(b) As $y \uparrow 1$, $F(y) \to 0.19 + 0.48 = 0.67$, and $F(1) = 1$. So $P(L = 1) = 1 - 0.67 = 0.33$.

(c) By Definition 4.1, $P(L \le 0.20) = F(0.20) = 0.19 + 0.48(0.20)^2 = 0.2092$. This includes the point mass at 0.

(d) $P(0.20 < L \le 0.85) = F(0.85) - F(0.20) = 0.48(0.85^2 - 0.20^2) = 0.3276$.

(e) Definition 4.4: $\phi_{.5}$ is the smallest y with $F(y) \ge .5$. Here $F(0) = 0.19 < .5$, so the median lies on the continuous part: $0.19 + 0.48\phi^2 = .5$, giving $\phi_{.5} = \sqrt{0.31/0.48} = 0.8036$.

(f) For every $y < 1$, $F(y) < 0.67 < .9$; F first reaches .9 by jumping to 1 at $y = 1$. So $\phi_{.9} = 1$: there is no y with $F(y) = .9$ exactly, which is why Definition 4.4 asks for the smallest y with $F(y) \geq p$. In words, the worst tenth of defaults are total losses.

Problem 19

seed 20260925

A corporate client of a Baku bank has a revolving credit line. Let Y be the fraction of the line in use at a month-end. The bank's model is the triangular density with peak at $m = 0.37$:

$$f(y) = \begin{cases} \dfrac{2y}{0.37}, & 0 \leq y \leq 0.37, \\ \dfrac{2(1-y)}{0.63}, & 0.37 < y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) For $0.37 < y \leq 1$: $F(y) =$ ________________ (a formula in y)

(b) The median utilisation $\phi_{.5}$: ________________

(c) $\phi_{0.19}$ (only 0.19 of month-ends have lower utilisation): ________________

(d) $\phi_{0.89}$ (the bank's "high-usage" alert level): ________________

For $0 \leq y \leq 0.37$, $F(y) = \int_0^y 2t/0.37\,dt = y^2/0.37$, so $F(0.37) = 0.37$: the peak sits at the 0.37-quantile.

(a) For $0.37 < y \leq 1$, it is easiest to integrate from the top:

$$1 - F(y) = \int_y^1 \frac{2(1-t)}{0.63}\,dt = \frac{(1-y)^2}{0.63} \quad \Longrightarrow \quad F(y) = 1 - \frac{(1-y)^2}{0.63}.$$

For each quantile, first decide which piece it lies on by comparing p with $F(0.37) = 0.37$, then solve $F(\phi_p) = p$ (Definition 4.4).

(b) $0.5 > 0.37$, so the median is on the right piece: $(1-\phi)^2/0.63 = 0.5$, $\phi_{.5} = 1 - \sqrt{0.5 \times 0.63} = 0.4388$.

(c) $0.19 < 0.37$, so use the left piece: $\phi^2/0.37 = 0.19$, $\phi_{0.19} = \sqrt{0.19 \times 0.37} = 0.2651$.

(d) $0.89 > 0.37$, so use the right piece: $(1-\phi)^2/0.63 = 1 - 0.89 = 0.11$, so $\phi_{0.89} = 1 - \sqrt{0.11 \times 0.63} = 0.7368$
.

Problem 20

seed 20260925

A Baku bank sends a customer's USD transfer abroad through a chain of correspondent banks. The time Y (hours) until the beneficiary is credited is never more than 8 hours, and the bank models it by

$$f(y) = \begin{cases} k\,y^2(8-y), & 0 \leq y \leq 8, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $k =$ ____________

(b) For $0 \le y \le 8$: $F(y) =$ ____________ (a formula in y)

(c) A customer calls after 3.2 hours: the money has not arrived. The probability it is still not credited after 5.2 hours: $P(Y > 5.2 \mid Y > 3.2) =$ ____________

(d) The bank wants to quote a time by which 0.80 of transfers are credited, the quantile $\phi_{0.80}$: ____________ hours

(a) On $[0, 8]$, $y^2(8 - y) \ge 0$, so Theorem 4.2 needs only

$$1 = k \int_0^8 (8y^2 - y^3)\, dy = k \left(\frac{8^4}{3} - \frac{8^4}{4}\right) = k \, \frac{8^4}{12} \implies k = \frac{12}{4096} = 0.00292969.$$

(b) For $0 \le y \le 8$,

$$F(y) = \frac{12}{4096} \left(\frac{8\,y^3}{3} - \frac{y^4}{4}\right) = \frac{32\,y^3 - 3y^4}{4096} = 4\left(\frac{y}{8}\right)^3 - 3\left(\frac{y}{8}\right)^4.$$

(c) Write $r = y/8$, so $F = 4r^3 - 3r^4$. At $y = 3.2$, $r = 0.4$ and $F = 0.179200$; at $y = 5.2$, $r = 0.65$ and $F = 0.562981$. Then

$$P(Y > 5.2 \mid Y > 3.2) = \frac{1 - F(5.2)}{1 - F(3.2)} = \frac{0.437019}{0.820800} = 0.5324.$$

(d) By Definition 4.4, $F(\phi) = 0.80$, i.e. with $x = \phi/8$,

$$4x^3 - 3x^4 = 0.80, \qquad 0 < x < 1.$$

F is increasing on $[0, 8]$, so there is one root; bisection gives $x = 0.78768$, hence $\phi_{0.80} = 8 \times 0.78768 = 6.3015$ hours.

Answer key

Problem	Answers
1	(a) 0, (b) 0.74, (c) 0.96, (d) 0.86, (e) 0.1
2	(a) 0.171633, (b) 0.188367, (c) 0.64, (d) 0
3	(a) 0.00554017, (b) 0.2025
4	(a) 0.949104, (b) 0.949104, (c) 0.00828723
5	(a) Discrete, (b) 0.66, (c) 0.82
6	(a) 2*(1-y/19)^1/19, (b) 0.687154, (c) 5.56497
7	(a) 0.0280702, (b) (y^2/2+1*y)/35.625, (c) 0.755263
8	(a) 0.00154321, (b) (72*y-y^2)/1296, (c) 0.229167, (d) 0.111111
9	(a) 2.25, (b) 0.311594, (c) 0.45975
10	(a) 0.769231, (b) 0.311538, (c) 0.553846, (d) 0.307692
11	(a) 0.125, (b) 0.59375, (c) 0.25

12	(a) 0.931392, (b) 0.225865, (c) 0.774135
13	(a) 0.04, (b) 2*y/25, (c) 0.6896, (d) 0.977324
14	(a) 0.21, (b) 0.21, (c) 0.5, (d) 8, (e) 20
15	(a) 1.83162, (b) 2*y/4.84*e^(-y^2/4.84), (c) 0.628213, (d) 0.559625
16	(a) 1-5/y, (b) 0.333333, (c) 0.652174, (d) 10
17	(a) 1.2, (b) 0.2, (c) 0.2+0.2*y+0.6*y^2, (d) 0.544, (e) 0.177132
18	(a) 0.19, (b) 0.33, (c) 0.2092, (d) 0.3276, (e) 0.803638, (f) 1
19	(a) 1-(1-y)^2/0.63, (b) 0.438751, (c) 0.265141, (d) 0.736751
20	(a) 0.00292969, (b) (32*y^3-3*y^4)/4096, (c) 0.53243, (d) 6.30146