Week 9, Problem Set 2 - worked solutions

Wackerly 4.3-4.4 - Expected Values for Continuous Random Variables; the Uniform Distribution

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A state bus operator in Baku tenders a one-year diesel-supply contract. From past tenders, the procurement office models the lowest bid Y (in thousands of AZN) as uniformly distributed between $\theta_1 = 195$ and $\theta_2 = 255$.

Give every answer to at least four significant figures.

(a) The probability that the lowest bid on the next tender is below 207.00 thousand AZN: ______________

(b) The probability that the lowest bid exceeds 246.00 thousand AZN: ______________

(c) $E(Y) =$ ______________ thousand AZN

(d) $V(Y) =$ ______________ (thousand AZN)^2

By Definition 4.6 the density is constant on $[\theta_1, \theta_2]$:

$$f(y) = \frac{1}{\theta_2 - \theta_1} = \frac{1}{255 - 195} = \frac{1}{60}, \qquad 195 \le y \le 255,$$

and zero elsewhere. A probability is the area under f, and under a flat density that area is (length of the subinterval)/(length of the whole interval).

(a) $P(Y < 207.00) = \int_{195}^{207.00} \frac{1}{60}\, dy = \frac{207.00-195}{60} = \frac{12.00}{60} = 0.2000$.

(b) $P(Y > 246.00) = \frac{255-246.00}{60} = \frac{9.00}{60} = 0.1500$.

(c) Theorem 4.6: $E(Y) = \frac{\theta_1+\theta_2}{2} = \frac{195+255}{2} = 225.00$ thousand AZN, the midpoint of the range.

(d) Theorem 4.6: $V(Y) = \frac{(\theta_2-\theta_1)^2}{12} = \frac{60^2}{12} = 300.0000$.

Notice that the variance depends only on the width of the bid range, not on where it sits: a range of 60 thousand AZN starting at 180 or at 260 is equally uncertain.

Problem 2

seed 20260925

The daily change Y in the price of a short-term Central Bank note, measured in qapik per 100 AZN of face value, is modelled by the density

$$f(y) = \begin{cases} k, & -34 \le y \le 34, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every number to at least four significant figures.

(a) $k =$ ______________

(b) For $-34 \le y \le 34$, the distribution function is $F(y) =$ ______________ (enter a formula in y)

(c) $F(13) = P(Y \le 13) =$ ______________

(d) The probability that the price rises by more than 7 qapik in a day: ______________

(e) $V(Y) =$ ______________ (qapik)^2

(a) A density must integrate to 1, so $\int_{-34}^{34} k\,dy = k(68) = 1$ and $k = 1/68 = 0.014706$. This is Definition 4.6 with $\theta_1 = -34$, $\theta_2 = 34$.

(b) For $-34 \le y \le 34$,

$$F(y) = \int_{-34}^{y} \frac{1}{68}\,dt = \frac{y+34}{68},$$

with $F(y) = 0$ below -34 and $F(y) = 1$ above 34.

(c) $F(13) = (13+34)/68 = 47/68 = 0.6912$.

(d) $P(Y > 7) = 1 - F(7) = \frac{34-7}{68} = \frac{27}{68} = 0.3971$.

(e) Theorem 4.6: $V(Y) = (\theta_2 - \theta_1)^2/12 = (68)^2/12 = 34^2/3 = 385.3333$, so the standard deviation is $\sigma = 19.6299$ qapik. The mean is $(\theta_1 + \theta_2)/2 = 0$: the model says the note is equally likely to rise or fall, and it is the width of the band that measures its risk.

Problem 3

seed 20260925

Let Y be the largest single cash withdrawal (in hundreds of thousands of AZN) from the vault of a regional bank branch on a given day. The branch's liquidity model uses the density

$$f(y) = \begin{cases} \dfrac{3y^2}{3.4^3}, & 0 \le y \le 3.4, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________

(b) $E(Y^2) =$ ______________

(c) $V(Y) =$ ______________

(d) $P(Y \le 2.890) =$ ______________

Write $\theta = 3.4$, so $\theta^3 = 39.304$.

(a) Definition 4.5:

$$E(Y) = \int_0^{\theta} y \cdot \frac{3y^2}{\theta^3}\,dy = \frac{3}{\theta^3} \cdot \frac{\theta^4}{4} = \frac{3\theta}{4} = 2.5500.$$

(b) Theorem 4.4 with $g(Y) = Y^2$:

$$E(Y^2) = \int_0^{\theta} y^2 \cdot \frac{3y^2}{\theta^3}\,dy = \frac{3}{\theta^3} \cdot \frac{\theta^5}{5} = \frac{3\theta^2}{5} = 6.9360.$$

(c) $V(Y) = E(Y^2) - [E(Y)]^2 = 6.9360 - 6.5025 = 0.4335$ (in general $3\theta^2/80$), so $\sigma = 0.6584$ hundred thousand AZN.

(d)

$$P(Y \le 2.890) = \int_0^{2.890} \frac{3y^2}{\theta^3}\,dy = \left(\frac{2.890}{\theta}\right)^3 = 0.85^3 = 0.6141.$$

This is the book's Example 4.6 stretched to a general range: the density rises towards the top of the range, so the mean sits three quarters of the way up, not in the middle.

Problem 4

seed 20260925

Large block trades in a bank's shares arrive on the Baku Stock Exchange according to a Poisson distribution. The trading session lasts 240 minutes, and on one particular day exactly one block trade was reported. Let Y be the time (in minutes after the open) at which it took place.

Give every probability to at least four significant figures.

(a) The probability that the block trade took place during the last 60 minutes of the session: ____________

(b) The probability that it took place between minute 55 and minute 100 of the session: ____________

(c) $E(Y) =$ ____________ minutes

Section 4.4 states the key fact: if the number of events in $(0, t)$ has a Poisson distribution and exactly one event is known to have occurred, its time of occurrence is uniformly distributed over $(0, t)$. So $Y \sim$ uniform on $(\theta_1, \theta_2) = (0, 240)$, with density $f(y) = 1/240$ there.

(a)

$$P(180 \le Y \le 240) = \int_{180}^{240} \frac{1}{240}\,dy = \frac{60}{240} = 0.2500.$$

(b)

$$P(55 \le Y \le 100) = \frac{100 - 55}{240} = \frac{45}{240} = 0.1875.$$

(c) Theorem 4.6: $E(Y) = (0 + 240)/2 = 120$ minutes.

As in Example 4.7, only the length of a window matters, not its position: the closing 60 minutes are no more likely to contain the trade than any other 60 minutes of the session. (Real block trades do cluster near the close; that would be evidence against the Poisson model, not a property of it.)

Problem 5

seed 20260925

Let Y be the proportion of a day during which a bank's ATM network is fully operational. The operations team fits the density

$$f(y) = \begin{cases} 4\,y^3, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The network's daily contribution to profit (in thousands of AZN) is $X = 115Y - 13$.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ____________

(b) $V(Y) =$ ____________

(c) $E(X) =$ ____________ thousand AZN

(d) $V(X) =$ _______________ (thousand AZN)^2

(a) Definition 4.5:

$$E(Y) = \int_0^1 y \cdot 4y^3 \, dy = 4 \cdot \frac{1}{5} = \frac{4}{5} = 0.800000.$$

(b) Theorem 4.4 with $g(Y) = Y^2$:

$$E(Y^2) = \int_0^1 y^2 \cdot 4y^3 \, dy = \frac{4}{6} = 0.666667,$$

so $V(Y) = E(Y^2) - [E(Y)]^2 = 0.666667 - 0.640000 = 0.026667$.

(c) Theorem 4.5 (parts 1-3) gives $E(aY - b) = aE(Y) - b$: $E(X) = 115(0.800000) - 13 = 79.0000$ thousand AZN.

(d) The constant shifts the distribution but does not spread it, and the factor a scales every deviation from the mean, so $V(X) = a^2 V(Y) = 13225(0.026667) = 352.6667$ (Exercise 4.26).

No density for X was needed: the linearity of expectation carries the moments of Y straight over to the profit.

Problem 6

seed 20260925

A chain of utility-payment kiosks in Baku does not give small change: every bill is rounded **up** to the next multiple of 5 qapik, and the kiosk keeps the difference. Let Y be the amount (in qapik) kept on one bill. Because the qapik part of a bill is effectively arbitrary, Y is modelled as uniformly distributed on $(0, 5)$.

Give every answer to at least four significant figures.

(a) $E(Y) =$ _______________ qapik

(b) $V(Y) =$ _______________ (qapik)^2

(c) The probability that more than 4.5 qapik is kept on a bill: _______________

(d) A kiosk processes 16500 bills a month. The expected total amount it keeps in a month, in AZN (100 qapik = 1 AZN):

____

Y is uniform on $(\theta_1, \theta_2) = (0, 5)$, so $f(y) = 1/5$ there (Definition 4.6).

(a) Theorem 4.6: $E(Y) = (0 + 5)/2 = 2.5$ qapik.

(b) Theorem 4.6: $V(Y) = 5^2/12 = 25/12 = 2.0833$, so $\sigma = 1.4434$ qapik.

(c) $P(Y > 4.5) = (5 - 4.5)/5 = 0.5/5 = 0.1000$.

(d) Let $Y_1, \ldots, Y_{16500}$ be the amounts kept on the month's bills. By Theorem 4.5 (part 3, the expectation of a sum is the sum of the expectations), $E(Y_1 + \cdots + Y_{16500}) = 16500 \times 2.5 = 41250.0$ qapik $= 412.50$ AZN. No independence is needed for this step.

Had the kiosk rounded to the **nearest** multiple of 5 qapik instead, the error would be uniform on $(-5/2, 5/2)$: the same width, hence the same variance 2.0833, but mean zero, so on average nobody would gain. This is why rounding error in measurement is usually modelled as uniform and centred.

Problem 7

seed 20260925

A bank's SME loans that default are pursued through a workout unit. For a defaulted loan with exposure 190 thousand AZN, the proportion Y of the exposure that is lost (the loss given default) has density

$$f(y) = \begin{cases} 0.50\,y + 2.25\,y^2, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The net amount the bank recovers, after a workout cost of 3.00 thousand AZN, is $W = 190(1 - Y) - 3.00$ thousand AZN.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ____________

(b) $E(Y^2) =$ ____________

(c) $V(Y) =$ ____________

(d) $E(W) =$ ____________ thousand AZN

(e) $V(W) =$ ____________ (thousand AZN)^2

First check that f is a density: it is non-negative on $[0, 1]$ and $\int_0^1 (0.50y + 2.25y^2)\,dy = 0.50/2 + 2.25/3 = 1$.

(a) Definition 4.5:

$$E(Y) = \int_0^1 y(0.50y + 2.25y^2)\,dy = \frac{0.50}{3} + \frac{2.25}{4} = 0.166667 + 0.562500 = 0.729167.$$

(b) Theorem 4.4 with $g(Y) = Y^2$:

$$E(Y^2) = \int_0^1 y^2(0.50y + 2.25y^2)\,dy = \frac{0.50}{4} + \frac{2.25}{5} = 0.125000 + 0.450000 = 0.575000.$$

(c) $V(Y) = E(Y^2) - [E(Y)]^2 = 0.575000 - 0.531684 = 0.043316$.

(d) Rewrite $W = (190 - 3.00) - 190Y = 187.00 - 190Y$. By Theorem 4.5,
$E(W) = 187.00 - 190\,E(Y) = 187.00 - 190(0.729167) = 48.4583$ thousand AZN.

(e) The constant 187.00 does not affect spread and the factor -190 enters squared:
$V(W) = 190^2 V(Y) = 36100(0.043316) = 1563.7066$.

The workout cost lowers the expected recovery one-for-one but leaves its variance untouched; only the size of the exposure scales the risk. This is the same calculation as Exercise 4.27, with a loan in place of the ore sample.

Problem 8

seed 20260925

A wholesaler sells to retailers on 34-day payment terms. Let Y be the number of days after delivery at which a retailer pays. Most retailers pay soon and few wait until the deadline, and the credit department models

$$f(y) = \begin{cases} \dfrac{2}{34}\left(1 - \dfrac{y}{34}\right), & 0 \le y \le 34, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $E(Y) =$ ____________ days

(b) $E(Y^2) =$ ______________

(c) $V(Y) =$ ______________

(d) The probability that a retailer takes more than 25.50 days to pay: ______________

Write $\theta = 34$, so $f(y) = \frac{2}{\theta} - \frac{2y}{\theta^2}$ on $[0, \theta]$.

(a) Definition 4.5:

$$E(Y) = \int_0^\theta \left(\frac{2y}{\theta} - \frac{2y^2}{\theta^2}\right) dy = \theta - \frac{2\theta}{3} = \frac{\theta}{3} = 11.3333.$$

(b) Theorem 4.4 with $g(Y) = Y^2$:

$$E(Y^2) = \int_0^\theta \left(\frac{2y^2}{\theta} - \frac{2y^3}{\theta^2}\right) dy = \frac{2\theta^2}{3} - \frac{\theta^2}{2} = \frac{\theta^2}{6} = \frac{1156}{6} = 192.6667.$$

(c) $V(Y) = E(Y^2) - [E(Y)]^2 = 192.6667 - 128.4444 = 64.2222$ (in general $\theta^2/18$), a standard deviation of 8.0139 days.

(d) Integrating the density over the upper tail, with the substitution $u = 1 - y/\theta$,

$$P(Y > 25.50) = \int_{25.50}^\theta \frac{2}{\theta}\left(1 - \frac{y}{\theta}\right) dy = \left(1 - \frac{25.50}{\theta}\right)^2 = 0.25^2 = 0.0625.$$

The mean payment time is a third of the term, not half, because the density puts more mass on early payment; a model that assumed uniform payment over the term would overstate the wholesaler's average receivables by half.

Problem 9

seed 20260925

The proportion Y of the business day during which every teller at a busy bank branch is serving a customer has density

$$f(y) = \begin{cases} c\,y^1(1-y)^5, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The branch prices the cost of customers queueing as $C = 360 + 800\,Y$ AZN per day.

You may use, for non-negative integers r and s,

$$\int_0^1 y^r(1-y)^s\,dy = \frac{r!\,s!}{(r+s+1)!}.$$

Give every answer to at least four significant figures.

(a) $c =$ ______________

(b) $E(Y) =$ ______________

(c) $V(Y) =$ ______________

(d) $E(C) =$ ______________ AZN

The integral identity can be checked by expanding $(1-y)^s$ with the binomial theorem and integrating term by term; it is the beta integral that returns in Section 4.7.

(a) The density must integrate to 1:

$$c\int_0^1 y^1(1-y)^5\,dy = c\cdot\frac{1!\,5!}{7!} = 1 \quad\Longrightarrow\quad c = \frac{7!}{1!\,5!} = \frac{5040}{1\times 120} = 42.$$

(b) Definition 4.5 raises the power of y by one:

$$E(Y) = c\int_0^1 y^2(1-y)^5\,dy = \frac{7!}{1!\,5!}\cdot\frac{2!\,5!}{8!} = \frac{2}{8} = 0.250000.$$

(c) Theorem 4.4 with $g(Y) = Y^2$ raises it by two:

$$E(Y^2) = \frac{7!}{1!\,5!}\cdot\frac{3!\,5!}{9!} = \frac{2\cdot 3}{8\cdot 9} = \frac{6}{72} = 0.083333,$$

so $V(Y) = 0.083333 - 0.062500 = 0.020833$.

(d) Theorem 4.5: $E(C) = 360 + 800\,E(Y) = 360 + 800(0.250000) = 560.00$ AZN.

Note that the variance was not needed for (d): the expected cost of a linear cost function depends on Y only through its mean.

Problem 10

seed 20260925

A bank runs its overnight risk calculations on rented cloud servers. The number Y of server-hours used in a week has density

$$f(y) = \begin{cases} \dfrac{12}{28^4}\,y^2(28-y), & 0 \le y \le 28, \\ 0, & \text{elsewhere.} \end{cases}$$

Server time costs 29 AZN per hour, so the weekly bill is $C = 29Y$ AZN.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________ hours

(b) $V(Y) =$ ______________ hours^2

(c) $E(C) =$ ______________ AZN

(d) $V(C) =$ ______________ AZN^2

(e) The IT budget allows 649.60 AZN a week. The probability that the weekly bill exceeds the budget: ______________

Write $\theta = 28$ and $f(y) = \frac{12}{\theta^4}(\theta y^2 - y^3)$.

(a) Definition 4.5:

$$E(Y) = \frac{12}{\theta^4}\int_0^\theta (\theta y^3 - y^4)\,dy = \frac{12}{\theta^4}\left(\frac{\theta^5}{4} - \frac{\theta^5}{5}\right) = \frac{3\theta}{5} = 16.8000.$$

(b) Theorem 4.4:

$$E(Y^2) = \frac{12}{\theta^4}\int_0^\theta (\theta y^4 - y^5)\,dy = \frac{12}{\theta^4}\left(\frac{\theta^6}{5} - \frac{\theta^6}{6}\right) = \frac{2\theta^2}{5} = 313.6000,$$

so $V(Y) = 313.6000 - 282.2400 = 31.3600$ (in general $\theta^2/25$).

(c) Theorem 4.5: $E(C) = 29\,E(Y) = 487.20$ AZN.

(d) $V(C) = 29^2 V(Y) = 841(31.3600) = 26373.76$, a standard deviation of 162.40 AZN.

(e) $C > 649.60$ exactly when $Y > 649.60/29 = 22.40$ hours, which is $x = 0.80$ of θ. The distribution function on $[0,\theta]$ is

$$F(y) = \frac{12}{\theta^4}\left(\frac{\theta y^3}{3} - \frac{y^4}{4}\right) = 4\left(\frac{y}{\theta}\right)^3 - 3\left(\frac{y}{\theta}\right)^4,$$

so $F(22.40) = 4(0.80)^3 - 3(0.80)^4 = 0.819200$ and $P(C > 649.60) = 1 - 0.819200 = 0.1808$.

This is Exercise 4.32 in a new setting. The budget sits above the mean bill of 487.20 AZN, and the exact tail probability shows how rarely it is breached; part (c) alone could not have told you that.

Problem 11

seed 20260925

When the cash-dispensing module of one of a bank's ATMs fails, a replacement is ordered from the supplier. The delivery time Y (in days) is uniformly distributed between 1 and 7 days. The module costs 850 AZN, and the fees and customer goodwill lost while the machine is down grow faster than linearly with the delay, so the total cost of one failure is

$$C = 850 + 50\,Y^2 \quad \text{AZN.}$$

Give every answer to at least four significant figures.

(a) The probability that delivery takes longer than 4.00 days: ____________

(b) $E(Y) =$ ____________ days

(c) $E(Y^2) =$ ____________

(d) $E(C) =$ ____________ AZN

By Definition 4.6, $f(y) = 1/(7-1) = 1/6$ for $1 \le y \le 7$.

(a) $P(Y > 4.00) = (7 - 4.00)/6 = 3.00/6 = 0.5000$.

(b) Theorem 4.6: $E(Y) = (1+7)/2 = 4.00$.

(c) Theorem 4.4 with $g(Y) = Y^2$:

$$E(Y^2) = \int_1^7 \frac{y^2}{6}\,dy = \frac{7^3 - 1^3}{3 \cdot 6} = \frac{343 - 1}{18} = \frac{342}{18} = 19.0000.$$

(Equivalently, $E(Y^2) = V(Y) + [E(Y)]^2 = 6^2/12 + 4.00^2$.)

(d) Theorem 4.5: $E(C) = 850 + 50\,E(Y^2) = 850 + 50(19.0000) = 1800.00$ AZN.

Plugging the mean delay into the cost function would give $850 + 50(4.00)^2 = 1650.00$ AZN, which is too low. Because the cost is convex in the delay, $E(Y^2) > [E(Y)]^2$, and the gap is exactly $50\,V(Y)$: the uncertainty in delivery is itself a cost.

Problem 12

seed 20260925

A payment processor submits card transactions to the clearing house in batches. The time Y (in minutes) from submission until a batch is settled is uniformly distributed between 35 and 75 minutes.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ____________ minutes

(b) $V(Y) =$ ____________ minutes^2

(c) The probability that a batch takes more than 69.00 minutes to settle: ____________

(d) A batch submitted 45.00 minutes ago has still not settled. The probability that it will take more than 69.00 minutes in total: _______________

(e) For the batch in (d), the expected total settlement time: _______________ minutes

By Definition 4.6, $f(y) = 1/40$ on $[35, 75]$.

(a) Theorem 4.6: $E(Y) = (35 + 75)/2 = 55.00$.

(b) Theorem 4.6: $V(Y) = 40^2/12 = 1600/12 = 133.3333$.

(c) $P(Y > 69.00) = (75 - 69.00)/40 = 6.00/40 = 0.1500$.

(d) By the definition of conditional probability, and because $\{Y > 69.00\} \subset \{Y > 45.00\}$,

$$P(Y > 69.00 \mid Y > 45.00) = \frac{P(Y > 69.00)}{P(Y > 45.00)} = \frac{6.00/40}{30.00/40} = \frac{6.00}{30.00} = 0.2000.$$

(e) The calculation in (d) works for any threshold between 45.00 and 75: given $Y > 45.00$, the probability of each subinterval is its length divided by 30.00. So conditionally Y is uniform on $(45.00, 75)$, and by Theorem 4.6 its conditional mean is $(45.00 + 75)/2 = 60.000$ minutes.

Having already waited **raises** the probability of a long wait, from 0.1500 to 0.2000: a uniform settlement time is not memoryless. The window of possible settlement times shrinks, and the late end becomes a larger share of what is left.

Problem 13

seed 20260925

An insurer's property desk records claims above 500,000 AZN, which arrive according to a Poisson distribution. During one 70-day reporting period, exactly one such claim was reported. Let Y be the day (measured continuously from the start of the period) on which it was reported.

Give every answer to at least four significant figures.

(a) The probability that the claim came in during the first 19 days: _______________

(b) The probability that it came in between day 30 and day 44: _______________

(c) An auditor who has seen only the first 30 days of the file knows that no large claim arrived in them. Given that information, the probability that the claim came in between day 30 and day 44: _______________

(d) Given the information in (c), the expected day on which the claim was reported: _______________

Section 4.4: when a Poisson count over $(0, t)$ is known to equal one, the time of the event is uniform on $(0, t)$. Here Y is uniform on $(0, 70)$ with $f(y) = 1/70$.

(a) $P(0 \le Y \le 19) = 19/70 = 0.2714$.

(b) $P(30 \le Y \le 44) = 14/70 = 0.2000$.

(c) The event $\{30 < Y \le 44\}$ lies inside $\{Y > 30\}$, so

$$P(30 < Y \le 44 \mid Y > 30) = \frac{P(30 < Y \le 44)}{P(Y > 30)} = \frac{14/70}{40/70} = \frac{14}{40} = 0.3500.$$

(d) The same argument applies to every subinterval of $(30, 70)$: its conditional probability is its length over 40. So given $Y > 30$, Y is uniform on $(30, 70)$, and by Theorem 4.6 its conditional mean is $(30 + 70)/2 = 50.0$.

The information in (c) does not change the **shape** of the distribution, only its range: the uniform model survives conditioning on a claim-free stretch. The probability of the same 14-day window rises from 0.2000 to 0.3500 because it now competes with fewer remaining days.

Problem 14

seed 20260925

The Central Bank's deposit-auction platform opens at 10:00 and accepts bids for 15 minutes, then closes for 30 minutes while it clears them, then opens again for 15 minutes, and so on. A commercial bank's treasury system sends its bid at a time Y that is uniformly distributed over the first 105 minutes after 10:00. A bid that arrives while the platform is closed is queued and entered the moment the platform reopens.

Give every answer to at least four significant figures.

(a) The probability that the bid is accepted on arrival (the platform is open): ______________

(b) Given that the bid arrives while the platform is closed, its expected queueing time: ______________ minutes

(c) The (unconditional) expected queueing time of the bid: ______________ minutes

Y is uniform on $(0, 105)$, so $f(y) = 1/105$ there. Within $(0, 105)$ the platform is open on the intervals

[0, 15], [45, 60], [90, 105]

(total length 45 minutes) and closed on

[15, 45], [60, 90]

(total length 60 minutes).

(a) $P(\text{open}) = 45/105 = 0.4286$. The long-run fraction of time open is $15/45 = 0.3333$; the answer differs from it whenever 105 minutes is not a whole number of 45-minute cycles.

(b) Let $g(y)$ be the queueing time of a bid sent at y: zero while open, and (time of reopening) $-\, y$ while closed. A closed interval $[c, c+\ell]$ ends at the reopening time $c + 30$ (an interval shorter than 30 minutes was cut off by the end of the 105-minute window, but the platform still reopens at $c + 30$). On such an interval

$$\int_c^{c+\ell} (c + 30 - y)\, dy = 30\ell - \frac{\ell^2}{2}.$$

Adding this over the closed intervals gives 900.00. By Theorem 4.4, $E[g(Y)] = 900.00/105$, and dividing by $P(\text{closed}) = 60/105$,

$$E(\text{wait} \mid \text{closed}) = \frac{900.00}{60} = 15.0000.$$

(For a complete closed interval this is just $30/2$, the mean of a uniform wait on $(0, 30)$.)

(c) $E[g(Y)] = \frac{1}{105} \int_0^{105} g(y)\, dy = \frac{900.00}{105} = 8.5714$ minutes, which is also $P(\text{closed}) \times E(\text{wait} \mid \text{closed}) = 0.5714 \times 15.0000$.

Problem 15

seed 20260925

Among the borrowers of a consumer-credit company who miss an instalment's due date, the delay Y (in days) until they pay is uniformly distributed on $(0, 16)$. The company's penalty G (in AZN) depends on the delay through a grace period of 8 days:

$$G = g(Y) = \begin{cases} 0, & 0 < Y \le 8, \\ 11 + 4.5\,(Y - 8), & 8 < Y < 16. \end{cases}$$

Give every answer to at least four significant figures.

(a) The probability that a late borrower is charged a penalty: ____________

(b) $E(G) =$ ____________ AZN

(c) The expected penalty of a borrower who **is** charged one: ____________ AZN

By Definition 4.6, $f(y) = 1/16$ on $(0, 16)$.

(a) $P(Y > 8) = (16 - 8)/16 = 8/16 = 0.5000$.

(b) Theorem 4.4 applies to any function of Y, including one defined piece by piece. The first piece contributes nothing, so

$$E(G) = \int_8^{16} [11 + 4.5(y - 8)] \frac{1}{16}\, dy = \frac{1}{16}\left[11 \cdot 8 + 4.5 \cdot \frac{8^2}{2}\right] = \frac{88.00 + 144.00}{16} = 14.5000.$$

(c) Dividing by $P(Y > 8)$,

$$E(G \mid Y > 8) = \frac{E(G)}{P(Y > 8)} = 11 + 4.5 \cdot \frac{8}{2} = 29.0000.$$

This also follows from the fact that, given $Y > 8$, the delay is uniform on $(8, 16)$, so the days beyond the grace period average $8/2$.

Note that $E(G) \ne g(E(Y))$: the mean delay $16/2$ may or may not exceed the grace period, but the expected penalty is always positive because some borrowers pay very late. For a function that is not linear, Theorem 4.4 must be applied to g itself.

Problem 16

seed 20260925

The monthly return X (in percent) of a manat money-market fund is modelled by the density

$$f(x) = \begin{cases} 0.16, & -1 < x \le 0, \\ 0.16 + 1.36\,x, & 0 < x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

An investor holds 470 thousand AZN in the fund, so her gain for the month is $G = 470X/100$ thousand AZN.

Give every answer to at least four significant figures.

(a) $P(X > 0) =$ ____________

(b) $E(X) =$ ____________

(c) $E(X^2) =$ ____________

(d) $V(X) =$ ____________

(e) $E(G) =$ ____________ thousand AZN

(f) The standard deviation of G: ____________ thousand AZN

Each integral splits at 0, where the formula for the density changes.

(a) $P(X \le 0) = \int_{-1}^{0} 0.16\,dx = 0.16$, so $P(X > 0) = 1 - 0.16 = 0.84$. (Check: $\int_0^1 (0.16 + 1.36x)\,dx = 0.16 + 0.68 = 0.84$.)

(b) Definition 4.5:

$$E(X) = \int_{-1}^{0} 0.16\,x\,dx + \int_{0}^{1} (0.16x + 1.36x^2)\,dx = -0.080 + 0.080 + \frac{1.36}{3} = 0.453333.$$

The flat piece on $(-1, 0]$ and the flat part of the density on $(0, 1]$ cancel; the mean comes entirely from the rising term.

(c) Theorem 4.4 with $g(X) = X^2$:

$$E(X^2) = \int_{-1}^{0} 0.16\,x^2\,dx + \int_{0}^{1} (0.16x^2 + 1.36x^3)\,dx = \frac{0.16}{3} + \frac{0.16}{3} + \frac{1.36}{4} = 2(0.053333) + 0.340000 = 0.446667.$$

(d) $V(X) = 0.446667 - 0.205511 = 0.241156$, so $\sigma_X = 0.491076$ percent.

(e) Theorem 4.5: $E(G) = 4.70\,E(X) = 4.70(0.453333) = 2.1307$.

(f) $V(G) = 4.70^2 V(X)$, so the standard deviation of G is $4.70 \times 0.491076 = 2.3081$ thousand AZN.

Problem 17

seed 20260925

A crude-oil tanker is due at the Sangachal terminal some time in a 9-hour window, and its arrival time Y (hours after the window opens) is uniformly distributed on $(0, 9)$. The terminal must schedule its berth crew to start at a fixed time t in the window.

- If the tanker arrives before t, it waits, and the terminal pays demurrage of 850 AZN per hour of waiting.
- If it arrives after t, the crew stands idle at a cost of 450 AZN per hour.

So the cost is $C_t = 850(t - Y)$ if $Y < t$ and $C_t = 450(Y - t)$ if $Y \ge t$.

Give every answer to at least four significant figures.

(a) The terminal currently schedules the crew for $t = 7.2$. Its expected cost $E(C_t)$ is ______________ AZN

(b) The start time t^* that minimises the expected cost: ______________ hours

(c) The minimum expected cost $E(C_{t^*})$: ______________ AZN

(d) The expected cost if the crew is scheduled for the **mean** arrival time $t = E(Y)$: ______________ AZN

Here $f(y) = 1/9$ on $(0, 9)$ (Definition 4.6), and $C_t = g(Y)$ for a function g with a kink at t. Theorem 4.4 splits the integral there:

$$E(C_t) = \frac{1}{9}\left[\int_0^t 850(t - y)\,dy + \int_t^9 450(y - t)\,dy\right] = \frac{850\,t^2 + 450\,(9 - t)^2}{2 \cdot 9}.$$

(a) At $t = 7.2$: $E(C_t) = \frac{850(7.2)^2 + 450(1.8)^2}{18} = \frac{44064.00 + 1458.00}{18} = 2529.00$ AZN.

(b) The expected cost is a quadratic in t. Setting its derivative to zero,

$$\frac{d}{dt}E(C_t) = \frac{2 \cdot 850\,t - 2 \cdot 450\,(9 - t)}{2 \cdot 9} = 0 \quad \Longrightarrow \quad t^* = \frac{450 \cdot 9}{850 + 450} = 3.1154.$$

The second derivative $(850 + 450)/9 > 0$, so this is a minimum. Equivalently, $P(Y < t^*) = t^*/9 = 450/1300$: schedule so that the chance of an early arrival equals the idle cost's share of the total.

(c) Substituting t^* gives $E(C_{t^*}) = \frac{850\cdot 450\cdot 9}{2(850+450)} = 1324.04$ AZN.

(d) At $t = E(Y) = 9/2$ (Theorem 4.6): $E(C_t) = \frac{(850+450)(9/2)^2}{2\cdot 9} = \frac{(850+450)\cdot 9}{8} = 1462.50$ AZN.

Scheduling for the mean arrival is optimal only when the two costs are equal. Otherwise the crew should be scheduled towards the side where being wrong is cheaper, which is why the answer to (d) is never below the answer to (c).

Problem 18

seed 20260925

An analyst models the price Y (in AZN) of a Baku-listed bank share on the expiry date of a one-month option as uniformly distributed between 24 and 53 AZN. A European call option with strike $K = 37.05$ AZN pays $C = \max(Y - 37.05, 0)$ at expiry; the put with the same strike pays $P = \max(37.05 - Y, 0)$.

Give every answer to at least four significant figures.

(a) The probability that the call finishes in the money ($Y > K$): ____________

(b) $E(C) =$ ____________ AZN

(c) $V(C) =$ ____________ AZN^2

(d) $E(P) =$ ____________ AZN

$f(y) = 1/29$ on $[24, 53]$ (Definition 4.6). The payoff $C = g(Y)$ is zero below the strike and $Y - K$ above it, so Theorem 4.4 integrates only over $[K, 53]$.

(a) $P(Y > 37.05) = (53 - 37.05)/29 = 15.95/29 = 0.5500$.

(b)

$$E(C) = \int_{37.05}^{53} (y - 37.05)\frac{1}{29}\,dy = \frac{(53 - 37.05)^2}{2\cdot 29} = \frac{15.95^2}{58} = 4.3863.$$

(c) Theorem 4.4 again, with $g(Y)^2$:

$$E(C^2) = \int_{37.05}^{53} (y - 37.05)^2\frac{1}{29}\,dy = \frac{15.95^3}{87} = 46.6405,$$

so $V(C) = E(C^2) - [E(C)]^2 = 46.6405 - 19.2392 = 27.4013$.

(d) By the same argument on $[24, K]$, $E(P) = \frac{(37.05-24)^2}{58} = \frac{13.05^2}{58} = 2.9362$.

Check: $C - P = Y - K$ for every Y, so by Theorem 4.5 $E(C) - E(P) = E(Y) - K = 38.50 - 37.05 = 1.4500$, which agrees with $4.3863 - 2.9362$. This is put-call parity in expectation. Note also that $E(C) \neq \max(E(Y) - K, 0)$: the option's value comes from the spread of Y, not only from its mean.

Problem 19

seed 20260925

Defaults in a pool of corporate loans held by a Baku bank occur according to a Poisson distribution. Over the next 5 years the bank's stress scenario assumes that **exactly one** default happens, at time Y (in years), with a loss of 2.2 million AZN at that moment. Discounting continuously at rate $\delta = 0.145$ per year, the present value of the loss is

$$W = 2.2\,e^{-0.145\,Y} \quad \text{million AZN.}$$

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________ years

(b) $E(W) =$ ______________ million AZN

(c) The present value obtained by discounting the loss from the **mean** default time, $2.2\,e^{-0.145\,E(Y)}$: ______________ million AZN

(d) $V(W) =$ ______________ (million AZN)^2

Section 4.4: given exactly one Poisson event in $(0, 5)$, its time is uniform on that interval, so $f(y) = 1/5$ for $0 \le y \le 5$. Write $\delta T = 0.145 \times 5 = 0.725$.

(a) Theorem 4.6: $E(Y) = 5/2 = 2.5$.

(b) Theorem 4.4 with $g(y) = Le^{-\delta y}$:

$$E(W) = \int_0^T Le^{-\delta y}\,\frac{1}{T}\,dy = \frac{L\,(1 - e^{-\delta T})}{\delta T} = \frac{2.2(1 - 0.484325)}{0.725} = 1.564808.$$

(c) $2.2\,e^{-0.3625} = 1.531055$.

(d) $W^2 = L^2 e^{-2\delta Y}$, so, by Theorem 4.4 again,

$$E(W^2) = \frac{L^2(1 - e^{-2\delta T})}{2\delta T} = \frac{4.84(1 - 0.234570)}{1.450} = 2.554952,$$

and $V(W) = 2.554952 - 2.448625 = 0.106327$.

The answer to (b) always exceeds the answer to (c). The discount factor $e^{-\delta y}$ is convex, so $E[g(Y)] \ne g(E(Y))$ and in this case $E[g(Y)] > g(E(Y))$ (Jensen's inequality): an early default costs more, in present value, than a late one saves. Plugging the mean default time into the discounting formula understates the expected loss.

Problem 20

seed 20260925

An insurer's actuaries model the size Y (in thousands of AZN) of a commercial fire claim by its distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - \left(1 - \dfrac{y}{190}\right)^2, & 0 \le y \le 190, \\ 1, & y > 190. \end{cases}$$

The insurer buys reinsurance that pays the part of each claim above a retention of 57.00 thousand AZN, that is, $R = \max(Y - 57.00, 0)$.

For a non-negative continuous Y you may use $E(Y) = \int_0^\infty [1 - F(y)]\,dy$ (Exercise 4.34). Give every number to at least four significant figures.

(a) For $0 \le y \le 190$, the density is $f(y) =$ ______________ (enter a formula in y)

(b) $E(Y) =$ ______________

(c) $E(Y^2) =$ ______________

(d) $V(Y) =$ ______________

(e) $E(R) =$ ______________ thousand AZN

Write $\theta = 190$ and $k = 2$.

(a) $f(y) = F'(y) = \frac{k}{\theta}\left(1 - \frac{y}{\theta}\right)^{k-1} = \frac{2}{190}\left(1 - \frac{y}{190}\right)^{1}$ on $[0, \theta]$, zero elsewhere.

(b) With the tail formula and the substitution $u = 1 - y/\theta$,

$$E(Y) = \int_0^\theta \left(1 - \frac{y}{\theta}\right)^k dy = \theta \int_0^1 u^k \, du = \frac{\theta}{k+1} = \frac{190}{3} = 63.3333.$$

(Definition 4.5 applied to the density from (a) gives the same value, with more work.)

(c) Theorem 4.4 with $g(Y) = Y^2$, and $y = \theta(1 - u)$:

$$E(Y^2) = \int_0^\theta y^2 \, \frac{k}{\theta}\left(1 - \frac{y}{\theta}\right)^{k-1} dy = k\theta^2 \int_0^1 (1-u)^2 u^{k-1} \, du = k\theta^2 \left[\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right] = \frac{2\theta^2}{(k+1)(k+2)}.$$

Here that is $2(36100)/12 = 6016.6667$.

(d) $V(Y) = E(Y^2) - [E(Y)]^2 = 6016.6667 - 4011.1111 = 2005.5556$.

(e) $R = g(Y)$ with $g(y) = 0$ for $y \le 57.00$ and $y - 57.00$ above. By Theorem 4.4, and integrating by parts (the boundary terms vanish because $y - 57.00 = 0$ at the retention and $1 - F(\theta) = 0$),

$$E(R) = \int_{57.00}^\theta (y - 57.00) f(y) \, dy = \int_{57.00}^\theta [1 - F(y)] \, dy = \frac{\theta}{k+1}\left(1 - \frac{57.00}{\theta}\right)^{k+1}.$$

Numerically, $\frac{190}{3}(0.70)^3 = 63.3333 \times 0.343000 = 21.7233$.

The reinsurer's expected payment is the area under the survival curve $1 - F$ to the right of the retention. As a share of the expected claim, $E(R)/E(Y) = (1 - 57.00/\theta)^{k+1} = 0.343000$: the retention keeps the remaining 0.657000 of the expected cost with the insurer, even though it is only a fraction 0.3 of the largest possible claim.

Answer key

Problem	Answers
1	(a) 0.2, (b) 0.15, (c) 225, (d) 300
2	(a) 0.0147059, (b) (y+34)/68, (c) 0.691176, (d) 0.397059, (e) 385.333
3	(a) 2.55, (b) 6.936, (c) 0.4335, (d) 0.614125
4	(a) 0.25, (b) 0.1875, (c) 120
5	(a) 0.8, (b) 0.0266667, (c) 79, (d) 352.667
6	(a) 2.5, (b) 2.08333, (c) 0.1, (d) 412.5
7	(a) 0.729167, (b) 0.575, (c) 0.043316, (d) 48.4583, (e) 1563.71
8	(a) 11.3333, (b) 192.667, (c) 64.2222, (d) 0.0625
9	(a) 42, (b) 0.25, (c) 0.0208333, (d) 560
10	(a) 16.8, (b) 31.36, (c) 487.2, (d) 26373.8, (e) 0.1808
11	(a) 0.5, (b) 4, (c) 19, (d) 1800
12	(a) 55, (b) 133.333, (c) 0.15, (d) 0.2, (e) 60
13	(a) 0.271429, (b) 0.2, (c) 0.35, (d) 50
14	(a) 0.428571, (b) 15, (c) 8.57143
15	(a) 0.5, (b) 14.5, (c) 29
16	(a) 0.84, (b) 0.453333, (c) 0.446667, (d) 0.241156, (e) 2.13067, (f) 2.30806

17	(a) 2529, (b) 3.11538, (c) 1324.04, (d) 1462.5
18	(a) 0.55, (b) 4.38625, (c) 27.4013, (d) 2.93625
19	(a) 2.5, (b) 1.56481, (c) 1.53106, (d) 0.106327
20	(a) 2*(1-y/190)^1/190, (b) 63.3333, (c) 6016.67, (d) 2005.56, (e) 21.7233