Week 10, Problem Set 1 - worked solutions

Wackerly 4.5 - The Normal Probability Distribution

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A bank's credit-scoring model reports every applicant's score in **standardised** form, so that the standardised score Z of a randomly chosen applicant is a standard normal random variable ($\mu = 0$, $\sigma = 1$).

Use Table 4, Appendix 3, and give each probability to four decimal places.

(a) $P(0 \le Z \le 0.73) =$ ______________

(b) The bank flags applicants whose score is unusually high for a manual fraud check: $P(Z > 2.34) =$

____

(c) $P(-0.79 \le Z \le 1.97) =$ ______________

(d) $P(-1.93 \le Z \le -0.58) =$ ______________

Table 4 gives $A(z) = P(Z > z)$, the area to the **right** of z for $z \ge 0$. Everything else follows from two facts: the total area is 1, and the density is symmetric about 0, so $P(Z < -z) = A(z)$ and $P(Z > 0) = 0.5$.

(a) The area from 0 to 0.73 is the right half minus the tail beyond 0.73:

$$P(0 \le Z \le 0.73) = 0.5 - A(0.73) = 0.5 - 0.2327 = 0.2673.$$

(b) This is a tabulated area directly: $P(Z > 2.34) = A(2.34) = 0.0096$.

(c) Remove both tails. By symmetry the left tail $P(Z < -0.79)$ equals $A(0.79)$:

$$P(-0.79 \le Z \le 1.97) = 1 - A(0.79) - A(1.97) = 1 - 0.2148 - 0.0244 = 0.7608.$$

(d) Reflect the interval to the positive side, where the table lives:

$$P(-1.93 \le Z \le -0.58) = P(0.58 \le Z \le 1.93) = A(0.58) - A(1.93) = 0.2810 - 0.0268 = 0.2542.$$

Because Z is continuous, $P(Z = z) = 0$, so it makes no difference whether the inequalities are strict.

Problem 2

seed 20260925

The monthly electricity bill Y of a household in one Baku district is approximately normally distributed with mean $\mu = 42$ AZN and standard deviation $\sigma = 12$ AZN.

Use Table 4 and give each answer to four decimal places.

(a) What fraction of households pay between 30 and 61.8 AZN?

____

(b) The distribution company sends an energy-saving letter to every household whose bill exceeds 62.4 AZN. What fraction of households receive the letter?

____

(c) What fraction of households pay less than 30 AZN?

____

Convert each bill to a z-score, its distance from the mean in standard deviations, $z = (y - \mu)/\sigma$. Then $Z = (Y - \mu)/\sigma$ is standard normal and Table 4 applies.

$$z_1 = \frac{30 - 42}{12} = -1.00, \qquad z_2 = \frac{61.8 - 42}{12} = 1.65, \qquad z_3 = \frac{62.4 - 42}{12} = 1.70.$$

(a) Remove the two tails. The left tail below -1.00 equals, by symmetry, $A(1.00) = 0.1587$; the right tail above 1.65 is $A(1.65) = 0.0495$. So

$$P(30 < Y < 61.8) = P(-1.00 < Z < 1.65) = 1 - 0.1587 - 0.0495 = 0.7919.$$

(b) $P(Y > 62.4) = P(Z > 1.70) = A(1.70) = 0.0446$.

(c) $P(Y < 30) = P(Z < -1.00) = A(1.00) = 0.1587$, which is the left tail already used in (a).

The same table serves every normal distribution because standardising moves any normal problem onto the one curve with $\mu = 0, \sigma = 1$.

Problem 3

seed 20260925

An energy analyst models the change Y (USD per barrel) in the Brent crude price over the coming month by the density

$$f(y) = \frac{1}{8\sqrt{2\pi}}\, e^{-(y+1.1)^2/128}, \qquad -\infty < y < \infty.$$

(a) $E(Y) =$ ________________

(b) $V(Y) =$ ________________ and the standard deviation of Y is ________________

(c) The median of Y, the value $\phi_{0.5}$ with $P(Y \leq \phi_{0.5}) = 0.5$, is ________________

(d) Using Table 4, find $P(Y > 7.54)$, the probability that the monthly change exceeds 7.54 USD per barrel, to four decimal places.

____

Compare with Definition 4.8, $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}$. The exponent is centred at $\mu = -1.1$ (note that $y + 1.1 = y - (-1.1)$), and the constant in front gives $\sigma = 8$. As a check, $2\sigma^2 = 2(8)^2 = 128$, which is the denominator in the exponent.

(a) By Theorem 4.7, $E(Y) = \mu = -1.1$ USD per barrel.

(b) Also by Theorem 4.7, $V(Y) = \sigma^2 = 64$, so the standard deviation is $\sigma = 8$.

(c) The normal density is symmetric about μ, so exactly half the area lies on each side of it: $P(Y \le \mu) = 0.5$ and the median equals the mean, -1.1.

(d) Standardise:

$$P(Y > 7.54) = P\left(Z > \frac{7.54 - (-1.1)}{8}\right) = P(Z > 1.08) = A(1.08) = 0.1401.$$

Problem 4

seed 20260925

A bank standardises the scores its credit model produces, so that the standardised score Z of a randomly chosen applicant has a standard normal distribution. The risk team sets its cut-offs directly on the Z scale.

Use Table 4. Table 4 lists z in steps of 0.01; an answer from either neighbouring table entry is accepted.

(a) A premium credit card is offered only to the highest-scoring applicants, a fraction 0.2119 of the whole pool. Find the cut-off z_0 with $P(Z > z_0) = 0.2119$.

$z_0 =$ ______________

(b) Find z_1 such that $P(Z < z_1) = 0.9535$.

$z_1 =$ ______________

(c) The "standard" band is the symmetric interval that contains the middle 0.90 of all scores. Find $z_2 > 0$ with $P(-z_2 < Z < z_2) = 0.90$.

$z_2 =$ ______________

Table 4 lists $A(z) = P(Z > z)$. An inverse problem looks the table up the other way: find the area in the body of the table and read off the z.

(a) The area to the right of z_0 is 0.2119, which is less than 0.5, so $z_0 > 0$. Scanning the body of Table 4 for 0.2119 gives $z_0 = 0.80$.

(b) Here $P(Z < z_1) = 0.9535 > 0.5$, so z_1 is positive and the area to its right is $1 - 0.9535 = 0.0465$. Table 4 shows $A(1.68) = 0.0465$, so $z_1 = 1.68$.

(c) The central area 0.90 leaves $1 - 0.90$ outside, split equally between the two tails by symmetry, so each tail has area $(1 - 0.90)/2 = 0.0500$. We need $A(z_2) = 0.0500$. Table 4 gives $A(1.64) = 0.0505$ and $A(1.65) = 0.0495$, which bracket 0.0500; either entry is accepted, and interpolation (or software, $qnorm$) gives

$z_2 \approx 1.645$.

Problem 5

seed 20260925

The weekly amount a bank branch spends on IT maintenance and repairs has been observed over a long period to be approximately normally distributed with mean 900 AZN and standard deviation 120 AZN. Next week's budget is 1050 AZN.

Use Table 4 and give each probability to four decimal places.

(a) What is the probability that next week's costs exceed the budget?

____

(b) What is the probability that next week's costs are below 678 AZN?

____

(c) Assume the costs in different weeks are independent and the budget stays at 1050 AZN each week. What is the probability that the budget is exceeded in none of the next 4 weeks?

____

Let Y be the weekly cost, $Y \sim$ normal with $\mu = 900$ and $\sigma = 120$, and standardise with $Z = (Y - \mu)/\sigma$.

(a)

$$P(Y > 1050) = P\left(Z > \frac{1050 - 900}{120}\right) = P(Z > 1.25) = A(1.25) = 0.1056.$$

(b)

$$P(Y < 678) = P\left(Z < \frac{678 - 900}{120}\right) = P(Z < -1.85) = A(1.85) = 0.0322,$$

using the symmetry of the normal density about 0.

(c) Each week the budget holds with probability $1 - 0.1056 = 0.8943$. Independence lets the probabilities multiply across weeks:

$$P(\text{no overrun in 4 weeks}) = (0.8943)^4 = 0.6398.$$

Even a budget that is rarely exceeded in a single week is exceeded somewhere in a run of weeks with noticeably higher probability.

Problem 6

seed 20260925

The monthly growth rate Y of retail deposits at a commercial bank, in percent, is approximately normal with mean $\mu = 0.6$ and standard deviation $\sigma = 1.9$.

(a) The empirical rule of Chapter 1 says that about 95% of a mound-shaped distribution lies within 2 standard deviation(s) of the mean. Give that interval $(\mu - 2\sigma,\ \mu + 2\sigma)$ for Y.

lower end = ______________ %, upper end = ______________ %

Now use Table 4 and give each probability to four decimal places.

(b) The exact normal probability that Y falls in the interval of (a):

____

(c) $P(-5.1 < Y < 4.4) =$ ______________

(d) The probability that deposits grow by more than 5.35 percent in a month:

____

(a) $\mu - 2\sigma = 0.6 - 2(1.9) = -3.2$ and $\mu + 2\sigma = 0.6 + 2(1.9) = 4.4$.

(b) On the z scale the interval is $(-2, 2)$ whatever μ and σ are, because $z = (y - \mu)/\sigma$ turns $\mu \pm k\sigma$ into $\pm k$:

$$P(\mu - 2\sigma < Y < \mu + 2\sigma) = P(-2 < Z < 2) = 1 - 2A(2.00) = 1 - 2(0.0228) = 0.9545.$$

This is the exact figure behind the empirical rule's "about 95%". The rule is the normal distribution's own $\pm 1, \pm 2, \pm 3$ areas (0.6826, 0.9544, 0.9974), rounded, which is why it works well for mound-shaped data and can fail badly for skewed data.

(c) The endpoints are 3 standard deviation(s) below and 2 above the mean: $(-5.1 - 0.6)/1.9 = -3$ and $(4.4 - 0.6)/1.9 = 2$. So

$$P(-3 < Z < 2) = 1 - A(3.00) - A(2.00) = 1 - 0.0013 - 0.0228 = 0.9759.$$

The empirical rule gives only a rough answer here, by taking half of each symmetric percentage; the table gives it directly.

(d) $z = (5.35 - 0.6)/1.9 = 2.50$, so $P(Y > 5.35) = A(2.50) = 0.0062$.

Problem 7

seed 20260925

An insurance company holds a securities portfolio worth 70 million AZN. Its risk department models the portfolio's daily return R, in percent, as normal with mean $\mu = 0.01$ and standard deviation $\sigma = 2.25$.

The one-day Value at Risk at confidence level 99.5% is the loss that is exceeded on only 0.005 of days: if r^* is the 0.005 quantile of R, that is $P(R \le r^*) = 0.005$, then $\text{VaR} = -r^* \times (\text{portfolio value})/100$.

Use Table 4; for quantiles an answer from either neighbouring table entry is accepted.

(a) Find the quantile r^* (in percent).

$r^* =$ ______________ %

(b) Find the one-day 99.5% VaR in million AZN.

VaR = ________________ million AZN

(c) What is the probability that tomorrow's loss exceeds 2,906,750 AZN? Give it to four decimal places.

____

(d) If the model is right, on how many of the next 250 trading days should the loss exceed the VaR of (b), on average?

____

(a) Standardise: $P(R \le r^*) = P\left(Z \le \frac{r^*-\mu}{\sigma}\right) = 0.005$. The point z with area 0.005 to its left is $-z_{0.005}$, where $A(z_{0.005}) = 0.005$. From Table 4 (between entries if necessary), $z_{0.005} \approx 2.576$. Hence

$$r^* = \mu - z_{0.005}\,\sigma = 0.01 - 2.576(2.25) = -5.786\%.$$

(b) A return of -5.786% on 70 million AZN is a loss of

$$\text{VaR} = \frac{-(-5.786)}{100} \times 70 = 4.050 \text{ million AZN.}$$

(c) A loss of 2,906,750 AZN on 70 million AZN is a return of $-2{,}906{,}750/(70 \times 10^6) \times 100 = -4.1525\%$. Then

$$P(R < -4.1525) = P\left(Z < \frac{-4.1525 - 0.01}{2.25}\right) = P(Z < -1.85) = A(1.85) = 0.0322.$$

(d) By construction the VaR is exceeded with probability 0.005 on each day, so over 250 days the expected number of exceedances is $250 \times 0.005 = 1.25$. Regulators compare the actual count with this figure when they backtest a bank's or insurer's VaR model.

Problem 8

seed 20260925

The total motor-insurance claims Y that an insurer pays in a week are approximately normal with mean 360 thousand AZN and standard deviation 58 thousand AZN.

Use Table 4. Answers from either neighbouring table entry are accepted.

(a) The insurer keeps a weekly liquidity reserve r and wants the claims to exceed it in only a fraction 0.1 of weeks, $P(Y > r) = 0.1$. How large must r be?

$r =$ ________________ thousand AZN

(b) Find the lower quartile $\phi_{0.25}$ of weekly claims, the value with $P(Y \le \phi_{0.25}) = 0.25$.

$\phi_{0.25} =$ ________________ thousand AZN

(c) Find the interquartile range $\phi_{0.75} - \phi_{0.25}$.

IQR = ______________ thousand AZN

Each part asks for a quantile of Y: find the z with the right area in Table 4, then undo the standardisation, $y = \mu + z\sigma$.

(a) We need $P(Z > (r-\mu)/\sigma) = 0.1$, so $(r-\mu)/\sigma = z_{0.1}$ with $A(z_{0.1}) = 0.1$. From Table 4, $z_{0.1} \approx 1.282$. Then

$$r = \mu + z_{0.1}\,\sigma = 360 + 1.282(58) = 434.33 \text{ thousand AZN.}$$

(b) The lower quartile has area 0.25 to its left, so its z-value is negative with $A(z) = 0.25$ on the positive side. Table 4 gives $A(0.67) = 0.2514$ and $A(0.68) = 0.2483$, so $z_{0.25} \approx 0.674$, and

$$\phi_{0.25} = \mu - 0.674\sigma = 360 - 0.674(58) = 320.88.$$

(c) By symmetry $\phi_{0.75} = \mu + 0.674\sigma = 399.12$, so

$$\phi_{0.75} - \phi_{0.25} = 2(0.674)\sigma = 1.35(58) = 78.24.$$

For every normal distribution the IQR is about 1.35σ. A sample whose IQR is far from 1.35 times its standard deviation is a first hint that the data are not normal.

Problem 9

seed 20260925

At a filling station, the amount Y of fuel actually dispensed when a customer pays for 20 litres is normally distributed with mean μ, which the operator sets by calibrating the pump, and standard deviation $\sigma = 0.38$ litres.

Use Table 4. Answers from either neighbouring table entry are accepted.

(a) The consumer-protection inspectorate requires that customers be short-changed ($Y < 20$) on at most 1% of fills. What is the smallest setting μ that meets the rule?

$\mu =$ ______________ litres

(b) With μ set as in (a), the pump gives away $\mu - 20$ litres per fill on average. The station makes 1100 such fills a day and fuel costs 1.00 AZN per litre. What is the expected daily cost of the overfill?

______________ AZN

(c) A new pump lets the operator adjust σ. What is the largest σ for which the amount dispensed lies within 0.54 litres of the mean with probability at least 0.99?

$\sigma =$ ______________ litres

(a) We need $P(Y < 20) = P\left(Z < \frac{20-\mu}{\sigma}\right) = 0.01$. The point with area 0.01 to its left is $-z_{0.01}$, where $A(z_{0.01}) = 0.01$; Table 4 gives $z_{0.01} \approx 2.326$. So

$$\frac{20-\mu}{0.38} = -2.326 \quad\Longrightarrow\quad \mu = 20 + 2.326(0.38) = 20.884 \text{ litres.}$$

Any larger μ also satisfies the rule, but gives away more fuel, so this is the smallest admissible setting.

(b) Expected overfill per fill is $E(Y) - 20 = \mu - 20 = 0.884$ litres (Theorem 4.7), so the daily cost is

$$1100 \times 0.884 \times 1.00 = 972.41 \text{ AZN}.$$

(c) $P(|Y - \mu| \leq 0.54) = P(-0.54/\sigma \leq Z \leq 0.54/\sigma) = 1 - 2A(0.54/\sigma)$. We need this to be at least 0.99, that is $A(0.54/\sigma) \leq 0.005$, which holds when $0.54/\sigma \geq z_{0.005} \approx 2.576$. Hence

$$\sigma \leq \frac{0.54}{2.576} = 0.2096 \text{ litres.}$$

A smaller σ would also meet the target; the largest one is what a cheaper pump can get away with.

Problem 10

seed 20260925

Two credit bureaus report scores on different scales. Among all adults in the country, bureau A's scores are approximately normal with mean 635 and standard deviation 100; bureau B's are approximately normal with mean 490 and standard deviation 120.

(a) A bank requires a bureau A score of at least 655. What fraction of adults score below 655 on bureau A? (Table 4, four decimal places.)

____

(b) For applicants who only have a bureau B score, the bank wants a **comparable** standard: a cut-off that excludes the same fraction of adults on bureau B. What cut-off should it use?

____

(c) Applicant X has a bureau A score of 805; applicant Y has a bureau B score of 754. Give each applicant's z-score within their own bureau, to two decimal places.

X: $z =$ _______________ Y: $z =$ _______________

(d) Relative to their own population, who has the stronger credit score?

- ○ Applicant X (bureau A)
- ○ Applicant Y (bureau B)
- ○ They are equally strong

(a) $z = (655 - 635)/100 = 0.20$. Since z is not negative,
$P(Z < 0.20) = 1 - A(0.20) = 1 - 0.4207 = 0.5793$.

(b) "Comparable" means the same position in its own distribution, that is the same z-score. The bureau B cut-off is therefore 0.20 standard deviations from bureau B's mean:

$$c_B = \mu_B + z\sigma_B = 490 + (0.20)(120) = 514.$$

It excludes exactly the fraction found in (a) because $P(Y_B < c_B) = P(Z < 0.20)$. No table look-up is needed for this part.

(c) $z_X = (805 - 635)/100 = 1.70$ and $z_Y = (754 - 490)/120 = 2.20$.

(d) The raw scores are on different scales and cannot be compared directly. The z-scores can: applicant Y sits further above (or less far below) the mean of their own population, so applicant Y is relatively stronger. Equivalently, a larger fraction of adults score below Y in their bureau than below the other applicant in theirs.

Problem 11

seed 20260925

A pension saver compares two funds. The annual return of the bond fund, R_A, is approximately normal with mean 3.6% and standard deviation 4%. The annual return of the equity fund, R_B, is approximately normal with mean 18.75% and standard deviation 25%.

Use Table 4 and give probabilities to four decimal places.

(a) Probability that each fund loses money over a year:

$P(R_A < 0) =$ ____________ $P(R_B < 0) =$ ____________

(b) The saver's target is a return above 8%:

$P(R_A > 8) =$ ____________ $P(R_B > 8) =$ ____________

(c) Which fund is less likely to lose money?

- ○ The bond fund
- ○ The equity fund

(d) Which fund is more likely to beat the 8% target?

- ○ The bond fund
- ○ The equity fund

(a) A loss is $R < 0$; standardise within each fund:

$$P(R_A < 0) = P\left(Z < \frac{0 - 3.6}{4}\right) = P(Z < -0.90) = A(0.90) = 0.1841,$$

$$P(R_B < 0) = P\left(Z < \frac{0 - 18.75}{25}\right) = P(Z < -0.75) = A(0.75) = 0.2266.$$

(b)

$$P(R_A > 8) = P\left(Z > \frac{8 - 3.6}{4}\right) = P(Z > 1.10) = A(1.10) = 0.1357,$$

$$P(R_B > 8) = P\left(Z > \frac{8 - 18.75}{25}\right) = P(Z > -0.43) = 1 - A(0.43) = 0.6664.$$

(c) The probability of a loss is $A(\mu/\sigma)$, which falls as μ/σ rises. $\mu_A/\sigma_A = 0.90$ and $\mu_B/\sigma_B = 0.75$, so the bond fund is less likely to lose money.

(d) The fund whose target z-score is smaller puts more area above the target: the equity fund is more likely to beat 8%.

Here the two criteria pick different funds. The chance of a loss depends only on mu/sigma, the chance of beating the target on (T - mu)/sigma, and a wide distribution can put more probability far above its mean than a narrow one with a better mu/sigma.

Problem 12

seed 20260925

An oil-trading company in Baku sells 20 thousand barrels of Brent crude forward, for delivery in three months at the fixed price $F = 80$ USD per barrel. Its analysts model the Brent spot price Y at delivery as normal with mean $\mu = 74$ and standard deviation $\sigma = 10$ USD per barrel. The profit and loss of the position, in USD, is

$$W = 20{,}000\,(F - Y) = -20{,}000\,Y + 20{,}000F.$$

A linear function $aY + b$ of a normal random variable is itself normal (this is proved in Chapter 6).

(a) $E(W) =$ ________________ USD

(b) The standard deviation of W is ________________ USD

Use Table 4 for (c) and (d), four decimal places.

(c) $P(W > 0)$, the probability that the forward sale makes a profit:

____

(d) The probability that the position loses more than 240,000 USD:

____

Write $W = aY + b$ with $a = -20{,}000$ and $b = 20{,}000F$.

(a) Expectation is linear, and $E(Y) = \mu$ by Theorem 4.7:

$$E(W) = aE(Y) + b = 20{,}000\,(F - \mu) = 20{,}000(6) = 120{,}000 \text{ USD}.$$

(b) $V(aY + b) = a^2V(Y)$, so the standard deviation is $|a|\sigma = 20{,}000 \times 10 = 200{,}000$ USD. The sign of a disappears: a short position is exactly as risky as a long one of the same size.

(c) $W > 0$ exactly when $Y < F$. Standardise Y: $z = (F - \mu)/\sigma = (80 - 74)/10 = 0.60$, so $P(Z < 0.60) = 1 - A(0.60) = 1 - 0.2743 = 0.7257$.

(d) $W < -240{,}000$ means $20{,}000(Y - F) > 240{,}000$, i.e. $Y > F + 240{,}000/20{,}000 = 92$. Then

$$P(Y > 92) = P\left(Z > \frac{92 - 74}{10}\right) = P(Z > 1.80) = A(1.80) = 0.0359.$$

Equivalently, W is normal with the mean and standard deviation of (a) and (b), and $(-240{,}000 - E(W))/\mathrm{SD}(W)$ gives the same z with the opposite sign: the lower tail of W is the upper tail of Y because $a < 0$.

Problem 13

seed 20260925

The credit scores S of applicants for a consumer loan at a bank are approximately normal with mean 620 and standard deviation 100. An applicant is approved if $S > 630$, and approved applicants with $S > 765$ are offered the bank's preferential interest rate.

Use Table 4; give probabilities to four decimal places.

(a) What proportion of applicants are approved?

____

(b) An applicant is known to have been approved. What is the probability that they are offered the preferential rate?

____

(c) Management considers replacing the rule by "offer the preferential rate to the top 15% of all applicants". What score cut-off would that be? (An answer from either neighbouring table entry is accepted.)

____

Standardise the two thresholds: $z_1 = (630 - 620)/100 = 0.10$ and $z_2 = (765 - 620)/100 = 1.45$.

(a) $P(S > 630) = P(Z > 0.10) = A(0.10) = 0.4602$.

(b) By the definition of conditional probability (Definition 2.9),

$$P(S > 765 \mid S > 630) = \frac{P(S > 765 \cap S > 630)}{P(S > 630)} = \frac{P(S > 765)}{P(S > 630)},$$

because every score above 765 is also above 630. The numerator is $A(1.45) = 0.0735$, so the probability is $0.0735/0.4602 = 0.1598$. Among approved applicants, the remaining 0.8402 get the standard rate.

Conditioning matters: among **all** applicants only 0.0735 qualify for the preferential rate, but approval has already removed the lower part of the distribution.

(c) We need c with $P(S > c) = 0.15$, i.e. $(c - 620)/100 = z_{0.15} \approx 1.036$ from Table 4. So

$$c = 620 + 1.036(100) = 723.64.$$

Problem 14

seed 20260925

A mobile operator finds that the monthly data use Y of its prepaid subscribers is approximately normal with mean 15.5 GB and standard deviation 5.8 GB. It uses quantiles of this distribution to design its tariffs.

Use Table 4. Answers from either neighbouring table entry are accepted.

(a) Find the data allowance C such that $P(Y < C) = 0.8643$.

$C =$ ______________ GB

(b) The heaviest 10% of users are to be offered an unlimited plan. Above what monthly use does a subscriber fall into that group?

______________ GB

(c) Find the interval, symmetric about the mean, that contains the middle 70% of subscribers' data use.

from ______________ GB to ______________ GB

Each part finds a z in Table 4 and converts back with $y = \mu + z\sigma$.

(a) $P(Y < C) = 0.8643$ means the area to the right of C is $1 - 0.8643 = 0.1357$, less than 0.5, so C is above the mean. Table 4 gives $A(1.10) = 0.1357$, so

$$C = 15.5 + 1.10(5.8) = 21.880 \text{ GB.}$$

(b) We need $P(Y > u) = 0.1$, so $(u - \mu)/\sigma = z_{0.1} \approx 1.282$ and

$$u = 15.5 + 1.282(5.8) = 22.93 \text{ GB.}$$

(c) The middle 70% leaves $(1 - 0.7)/2 = 0.15$ in each tail, by symmetry. Table 4 gives $z_{0.15} \approx 1.036$, so the interval is

$$\mu \pm 1.036\sigma = 15.5 \pm 1.036(5.8) = (9.49,\ 21.51) \text{ GB.}$$

Problem 15

seed 20260925

The daily return R (in percent) of a government Eurobond held by an Azerbaijani pension fund is approximately normal with mean 0.02 and standard deviation 0.85. The fund's risk report calls a day "bad" if $R < -1.1615$. Treat returns on different days as independent.

Use Table 4 and give answers to four decimal places.

(a) The probability that a given day is bad:

____

(b) The probability that none of the next 12 trading days is bad:

____

(c) The probability that at least two of the next 12 trading days are bad:

____

(d) The expected number of bad days among the next 12:

____

(a)

$$p = P(R < -1.1615) = P\left(Z < \frac{-1.1615 - 0.02}{0.85}\right) = P(Z < -1.39) = A(1.39) = 0.0823.$$

Now let X be the number of bad days among 12. Each day is bad with the same probability p, independently, so X is binomial with $n = 12$ and $p = 0.0823$ (Definition 3.7).

(b) $P(X = 0) = (1 - p)^{12} = (0.9177)^{12} = 0.3570$.

(c) $P(X = 1) = 12\,p\,(1 - p)^{11} = 0.3840$, so

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - 0.3570 - 0.3840 = 0.2591.$$

(d) $E(X) = np = 12(0.0823) = 0.9872$ (Theorem 3.7).

The normal distribution supplies the single-day probability; the binomial distribution of Chapter 3 then counts how often that event happens.

Problem 16

seed 20260925

A mint strikes coins whose nominal weight is 3.90 g. A coin is accepted only if its weight is in the interval 3.90 ± 0.0722 g, that is between 3.8278 and 3.9722 g; the rest are melted down. The weight of a coin from the current press setting is normal with mean $\mu = 3.8791$ g and standard deviation $\sigma = 0.038$ g.

Use Table 4; probabilities to four decimal places.

(a) With the current setting, what fraction of coins is rejected?

____

(b) σ cannot be changed, but μ can. Which mean weight minimises the fraction rejected?

$\mu =$ ________________ g

(c) What fraction is rejected at that setting?

____

(d) The mint strikes 750,000 coins a day. About how many fewer coins a day are rejected after the adjustment?

____

(a) Standardise both specification limits using the current mean:

$$z_{\text{upper}} = \frac{3.9722 - 3.8791}{0.038} = 2.45, \qquad z_{\text{lower}} = \frac{3.8278 - 3.8791}{0.038} = -1.35.$$

A coin is rejected if it is in either tail:

$$P(\text{reject}) = P(Z > 2.45) + P(Z < -1.35) = A(2.45) + A(1.35) = 0.0071 + 0.0885 = 0.0957.$$

The press runs light, so most rejects are in the light tail.

(b) The acceptance interval has fixed width $2d$. The probability that a normal variable lands in an interval of fixed width is largest when the interval is centred on μ, because the density is symmetric and highest at μ (moving the interval away from the centre swaps area near the peak for area in a tail). So set $\mu = 3.90$ g, the midpoint of the specification.

(c) Now both limits are $0.0722/0.038 = 1.90$ standard deviations away:

$$P(\text{reject}) = 2A(1.90) = 2(0.0287) = 0.0574.$$

(d) $750{,}000 \times (0.0957 - 0.0574) \approx 28663$ coins a day.

Problem 17

seed 20260925

A bank's analysts believe that the monthly card spending Y (AZN) of its premium cardholders is approximately normal, but they do not know μ or σ. What they do know from the card system is that a fraction 0.1020 of cardholders spend less than 557.4 AZN a month and a fraction 0.0322 spend more than 1743 AZN.

Use Table 4. Answers consistent with the neighbouring table entries are accepted.

(a) Find μ and σ.

$\mu =$ ______________ AZN $\sigma =$ ______________ AZN

(b) What fraction of premium cardholders spend more than 1211 AZN a month? (Four decimal places.)

____

(c) The bank sets each card's monthly limit at the 99th percentile of spending. What is that limit?

______________ AZN

(a) Each fact is a statement about a quantile, and each gives one linear equation in μ and σ.

The fraction below 557.4 is $0.1020 < 0.5$, so 557.4 lies below the mean. By symmetry $P(Z < -z) = A(z)$, and Table 4 shows $A(1.27) = 0.1020$; so $(557.4 - \mu)/\sigma = -1.27$.

The fraction above 1743 is 0.0322; Table 4 shows $A(1.85) = 0.0322$; so $(1743 - \mu)/\sigma = 1.85$.

That is

$$\mu - 1.27\,\sigma = 557.4, \qquad \mu + 1.85\,\sigma = 1743.$$

Subtract the first from the second: $(1.85 + 1.27)\sigma = 1743 - 557.4$, so

$$\sigma = \frac{1185.6}{3.12} = 380, \qquad \mu = 557.4 + 1.27(380) = 1040.$$

(b) $z = (1211 - 1040)/380 = 0.45$, so $P(Y > 1211) = P(Z > 0.45) = A(0.45) = 0.3264$.

(c) The 99th percentile has area 0.01 to its right: $z_{0.01} \approx 2.326$ (between the Table 4 entries 2.32 and 2.33), so

$$\phi_{0.99} = \mu + 2.326\sigma = 1040 + 2.326(380) = 1924.01 \text{ AZN}.$$

Only one cardholder in a hundred would hit the limit in a typical month.

Problem 18

seed 20260925

A bank's treasury runs a foreign-currency position of 20 million AZN. The position's daily return R (in percent) is modelled as normal with mean 0.04 and standard deviation 0.70. The daily loss is $L = -R \times 20/100$ million AZN. The board has set a limit: the one-day 99% Value at Risk, the loss exceeded with probability only 0.01, must not be more than 378,400 AZN.

Use Table 4; quantile answers consistent with either neighbouring table entry are accepted.

(a) Compute the one-day 99% VaR, in AZN.

_______________ AZN

(b) Is the limit breached?

- ◯ Yes: the 99% VaR is larger than the limit
- ◯ No: the 99% VaR is within the limit

(c) What is the probability that tomorrow's loss exceeds 378,400 AZN? (Four decimal places.)

____

(d) Keeping the position at 20 million AZN and the mean at 0.04%, what is the largest daily standard deviation (in %) compatible with the limit?

_______________ %

(e) Keeping $\sigma = 0.70$, what is the largest position (million AZN) compatible with the limit?

_______________ million AZN

(a) The 99% VaR is minus the 0.01 quantile of the return, scaled to money. The 0.01 quantile of R is $\mu - z_{0.01}\sigma$, with $z_{0.01} \approx 2.326$ (between the Table 4 entries 2.32 and 2.33). So

$$\text{VaR} = (z_{0.01}\sigma - \mu)\,\frac{V}{100} = (2.326 \times 0.70 - 0.04)\% \times 20 \text{ million} = 317{,}689 \text{ AZN.}$$

(b) The VaR is below the limit, so the desk is within its limit.

(c) A loss of 378,400 AZN is a return of $r = -378{,}400/(20 \times 10^6) \times 100 = -1.8920\%$, and

$$P(R < -1.8920) = P\left(Z < \frac{-1.8920 - 0.04}{0.70}\right) = P(Z < -2.76) = A(2.76) = 0.0029.$$

This agrees with (b): the limit is breached exactly when this probability is more than 0.01, i.e. when $2.76 < z_{0.01}$. Here 0.0029 is less than 0.01, consistent with (b).

(d) The limit expressed as a return is $378{,}400/(20 \times 10^4) = 1.8920\%$. We need $z_{0.01}\sigma - \mu \le 1.8920$, so

$$\sigma \le \frac{1.8920 + 0.04}{2.326} = 0.8305\%.$$

(e) The VaR is proportional to the position: $\text{VaR} = V \times (1.5884\%)$. Setting it equal to the limit,

$$V_{\text{max}} = \frac{378{,}400}{1.5884/100 \times 10^6} = 23.822 \text{ million AZN.}$$

Problem 19

seed 20260925

A risk analyst has 252 months of monthly returns of a broad government-bond index. She standardises each return using the sample mean and standard deviation, and counts the months on which the return was more than 2.95 standard deviations away from the mean, in either direction. She finds 2 of them.

Use Table 4.

(a) If the returns were normal, what is the probability that a single month's return lies more than 2.95 standard deviations from the mean? (Four decimal places.)

____

(b) How many such months would the normal model lead her to expect in the sample?

____

(c) What should she conclude?

- ◯ The count is close to what the normal model predicts; on this evidence there is no reason to doubt normality.
- ◯ There are several times more extreme returns than the normal model predicts: the tails are heavier than normal, so normality is doubtful.
- ◯ There are fewer extreme returns than predicted, so the returns must follow a uniform distribution.

(d) Table 4 also lists areas far in the tail. Under a normal model, a fall of more than 4.5 standard deviations below the mean happens on average once in how many months? (Enter $1/P(Z > 4.5)$; answers within 2% are accepted.)

____

(a) By symmetry,

$$P(|Z| > 2.95) = 2A(2.95) = 2(0.0016) = 0.0032.$$

(b) Each of the 252 months is such a move with probability 0.0032, so the expected number is $252 \times 0.0032 = 0.80$.

(c) The sample has 2 such months against about 0.80 expected, which is well within ordinary chance variation. Returns aggregated over a month and over many bonds are much closer to normal than daily returns on a single share, and this check gives no reason to reject the normal model. (One check passing does not prove normality; it only fails to contradict it.)

(d) Table 4 gives $A(4.5) = 0.00000340$, so such a fall is expected once in $1/0.00000340 \approx 294{,}118$ months, roughly 24,510 years. For a monthly bond index that is not unreasonable; but the same calculation for daily share returns gives waiting times that real markets contradict several times a decade.

Problem 20

seed 20260925

A bank's scoring model gives every loan applicant a score S. From past data, applicants who later default have scores that are normal with mean 595 and standard deviation 40; applicants who repay have scores that are normal with mean 659 and the same standard deviation. A fraction $\pi = 0.14$ of applicants would default if given a loan.

The bank approves every applicant whose score exceeds a cut-off c, and chooses c so that only a fraction 0.0099 of would-be defaulters are approved.

Use Table 4; give probabilities to four decimal places.

(a) Find the cut-off c. (An answer from either neighbouring table entry is accepted.)

$c =$ ________________

(b) What fraction of applicants who would repay are approved?

____

(c) What fraction of all applicants are approved?

____

(d) An applicant has just been approved. What is the probability that they will default?

____

Let S_B and S_G be the scores of a would-be defaulter and of a repayer.

(a) We need $P(S_B > c) = 0.0099$, so $(c - 595)/40 = z$ with $A(z) = 0.0099$. Table 4 gives $z = 2.33$, so

$$c = 595 + 2.33(40) = 688.2.$$

(b) Standardise the same cut-off in the repayers' distribution: $(c - 659)/40 = 0.73$, and $P(S_G > 688.2) = P(Z > 0.73) = A(0.73) = 0.2327$.

The two populations share σ, so all that matters is how far apart the means are in standard deviations, $(659 - 595)/40$. The further apart, the better a single cut-off separates them.

(c) By the law of total probability (Theorem 2.8), conditioning on the applicant's type,

$$P(\text{approved}) = \pi\, P(S_B > c) + (1 - \pi)\, P(S_G > c) = 0.14(0.0099) + 0.86(0.2327) = 0.2015.$$

(d) By Bayes' rule (Theorem 2.9),

$$P(\text{default} \mid \text{approved}) = \frac{\pi\, P(S_B > c)}{P(\text{approved})} = \frac{0.0014}{0.2015} = 0.0069.$$

Only 0.99% of would-be defaulters get through, but because the bank sees applicants of both kinds, the default rate **among approved loans** is 0.0069, which is the figure that drives the bank's credit losses.

Answer key

Problem	Answers
1	(a) 0.267305, (b) 0.00964187, (c) 0.760817, (d) 0.254154
2	(a) 0.791874, (b) 0.0445655, (c) 0.158655
3	(a) -1.1, (b) 64, (c) 8, (d) -1.1, (e) 0.140071
4	(a) 0.8, (b) 1.68, (c) 1.64485
5	(a) 0.10565, (b) 0.0321569, (c) 0.639779
6	(a) -3.2, (b) 4.4, (c) 0.9545, (d) 0.9759, (e) 0.00620967
7	(a) -5.78562, (b) 4.04993, (c) 0.0321569, (d) 1.25
8	(a) 434.33, (b) 320.88, (c) 78.2408
9	(a) 20.884, (b) 972.414, (c) 0.209641
10	(a) 0.57926, (b) 514, (c) 1.7, (d) 2.2, (e) Applicant Y (bureau B)
11	(a) 0.18406, (b) 0.226627, (c) 0.135666, (d) 0.666402, (e) The bond fund, (f) The equity fund
12	(a) 120000, (b) 200000, (c) 0.725747, (d) 0.0359304
13	(a) 0.460172, (b) 0.159786, (c) 723.643
14	(a) 21.88, (b) 22.933, (c) 9.48871, (d) 21.5113
15	(a) 0.0822643, (b) 0.356953, (c) 0.259086, (d) 0.987172
16	(a) 0.0956507, (b) 3.9, (c) 0.0574332, (d) 28663.1
17	(a) 1040, (b) 380, (c) 0.326355, (d) 1924.01
18	(a) 317689, (b) Choice 2, (c) 0.00289007, (d) 0.830486, (e) 23.822
19	(a) 0.00317774, (b) 0.80079, (c) The count ... normality., (d) 294319
20	(a) 688.2, (b) 0.232695, (c) 0.201504, (d) 0.00688041