

Week 10, Problem Set 2

Wackerly 4.6 - The Gamma, Exponential and Chi-Square Distributions

This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku brokerage records the time Y (in minutes) between consecutive large trades (single orders above 500,000 AZN) in its order book. Over a normal trading session Y has an exponential distribution (Definition 4.11) with $\beta = 12$ minutes.

Give every probability to at least four significant figures.

(a) The probability that the next gap exceeds 23 minutes: $P(Y > 23) =$ _____

(b) The probability that the next gap is at most 14 minutes: $P(Y \leq 14) =$ _____

(c) $P(14 < Y \leq 23) =$ _____

(d) The standard deviation of the gap, in minutes: $\sigma =$ _____

Problem 2

seed 20260925

A motor insurer in Baku models the waiting time Y (in days) between successive claims on one fleet policy as exponential with unknown mean β . Its claims history shows that the policy goes more than 6 days without a claim with probability

$$P(Y > 6) = 0.32.$$

Give every answer to at least four significant figures.

(a) $\beta = E(Y) =$ _____ days

(b) $P(Y \leq 6.6) =$ _____

(c) $V(Y) =$ _____ days^2

Problem 3

seed 20260925

The service time Y (in minutes) of a call to a bank's customer-support line has a gamma distribution (Definition 4.9) with $\alpha = 2.0$ and $\beta = 3.8$. Each call costs the bank a fixed 1.90 AZN for the telephony platform plus 0.35 AZN per minute of agent time, so the cost of a call is $C = 1.90 + 0.35 Y$ AZN.

Give every answer to at least four significant figures.

(a) $E(Y) =$ _____ minutes

(b) $V(Y) =$ _____

(c) $\sigma_Y =$ _____ minutes

(d) $E(C) =$ _____ AZN

(e) $V(C) =$ _____

Problem 4

seed 20260925

A bank's model-validation team backtests its Value-at-Risk model each quarter. When the model is correctly specified, the backtest statistic Y has a chi-square distribution with $\nu = 10$ degrees of freedom.

(a) $E(Y) =$ _____

(b) $V(Y) =$ _____

(c) By Definition 4.10, Y is a gamma random variable with $\beta = 2$ and $\alpha =$ _____

(d) For its dashboard the team reports the rescaled score $W = 4.65 Y$. W is again gamma distributed with the same α . Find $E(W) =$ _____

(e) $V(W) =$ _____

Give non-integer answers to at least four significant figures.

Problem 5

seed 20260925

A data centre in Sumgait buys solid-state drives whose lifetime Y (in months) is exponentially distributed with mean $\beta = 37$ months.

Give every answer to at least four significant figures.

(a) The median lifetime $\phi_{0.5}$, the value with $P(Y \leq \phi_{0.5}) = 0.5$: _____ months

(b) The supplier wants a warranty period w short enough that only 14% of drives fail before it, that is $P(Y \leq w) = 0.14$. $w =$ _____ months

(c) The probability that a drive fails within its first 18 months: $P(Y \leq 18) =$ _____

Problem 6

seed 20260925

Customers phoning a mobile operator's billing line wait on hold for a time Y (in minutes) that is exponentially distributed with mean $\beta = 4.0$ minutes. A customer has already been on hold for 10 minutes.

Give every probability to at least four significant figures.

(a) The probability that she is still on hold 3.5 minutes from now: $P(Y > 13.5 \mid Y > 10) =$ _____

(b) For a customer who has just joined the queue, $P(Y > 3.5) =$ _____

(c) For a customer who has just joined the queue, $P(Y > 13.5) =$ _____

(d) The expected **additional** hold time of the customer who has already waited 10 minutes,
 $E(Y - 10 \mid Y > 10) =$ _____ minutes

Problem 7

seed 20260925

During the afternoon session, large trades in a Baku brokerage's order book arrive so that the number of large trades in any t minutes has a Poisson distribution with mean λt , where $\lambda = 0.15$ per minute, and counts in non-overlapping periods are independent. Let W be the waiting time (in minutes) from 14:00 until the first large trade.

Give every number to at least four significant figures.

(a) The probability of no large trade in the first 10 minutes: _____

(b) The event $\{W > w\}$ is the event "no large trade in the first w minutes". Use this to find the distribution function of W for $w \geq 0$: $F(w) = P(W \leq w) =$ _____

(Enter a function of w ; write the exponential as $e^{...}$ or $\exp(...)$.)

(c) W is exponential (Definition 4.11). Its mean $\beta = E(W) =$ _____ minutes

(d) The median waiting time: _____ minutes

(e) $P(W \leq 2.0) =$ _____

(f) The probability of at least two large trades in the first 10 minutes: _____

Problem 8

seed 20260925

The time Y (in working days) that a Baku bank takes to process a mortgage application, from submission to a credit decision, has a gamma distribution with $\alpha = 2$ and $\beta = 6.0$.

Because α is a whole number, gamma probabilities can be written as sums of Poisson probabilities (Exercise 4.99): for $\lambda > 0$,

$$\frac{1}{\Gamma(\alpha)} \int_{\lambda}^{\infty} u^{\alpha-1} e^{-u} du = \sum_{x=0}^{\alpha-1} \frac{\lambda^x e^{-\lambda}}{x!}.$$

Give every answer to at least four significant figures.

(a) $E(Y) =$ _____ days

(b) The probability that a decision takes more than 14.4 working days: $P(Y > 14.4) =$ _____

(c) The probability that a decision arrives within 21.6 working days: $P(Y \leq 21.6) =$ _____

Problem 9

seed 20260925

When a cooling failure takes a server hall offline at a Baku data centre, the outage lasts Y hours, exponentially distributed with mean $\beta = 1.2$ hours. Penalties to clients grow faster than the outage itself, and the total cost (in thousands of AZN) is

$$C = 10 + 15Y + 5.5Y^2.$$

Give every answer to at least four significant figures.

(a) $E(Y^2) =$ _____

(b) $E(C) =$ _____ thousand AZN

(c) $V(C) =$ _____

(Hint: you will need $E(Y^3)$ and $E(Y^4)$.)

Problem 10

seed 20260925

A colocation provider models the lifetime Y (in months) of the power supply in a rack server as exponential with mean $\beta = 32$ months. An auditor finds a server whose power supply has already worked for 45 months.

Give every answer to at least four significant figures.

(a) The probability that this power supply is still working 18 months from now: $P(Y > 63 \mid Y > 45) =$ _____

(b) For comparison, the probability that a **new** power supply lasts 63 months: $P(Y > 63) =$ _____

(c) The expected total lifetime of the audited power supply, $E(Y \mid Y > 45) =$ _____ months

(d) The median total lifetime of the audited power supply, the value m with $P(Y > m \mid Y > 45) = 0.5$: $m =$ _____ months

Problem 11

seed 20260925

A bank's backup system stores its transaction archive on an array of 3 drives. Each drive sits on its own controller, with its own power feed, in a different rack, so the drive lifetimes are independent; each lifetime is exponential with mean $\beta = 6.5$ years. The array keeps the archive readable as long as at least 1 of the 3 drives work (it tolerates 2 failed drive(s)). No drives are replaced.

Give every answer to at least four significant figures.

(a) The probability that one particular drive is still working after 2.75 years: _____

(b) The probability that the array still keeps the archive readable after 2.75 years: _____

(c) The expected number of drives still working after 2.75 years: _____

Problem 12

seed 20260925

At a Baku insurer's claims call centre, the length Y (in minutes) of a call is exponential with mean $\beta = 4.0$ minutes. A call longer than 19 minutes is escalated to a supervisor. Consider the next 11 calls, whose lengths are independent.

Give every answer to at least four significant figures.

- (a) The probability that a single call is escalated: $P(Y > 19) =$ _____
- (b) The probability that at least one of the next 11 calls is escalated: _____
- (c) The probability that exactly two of the next 11 calls are escalated: _____
- (d) The expected number of escalations among the next 11 calls: _____

Problem 13

seed 20260925

The order-acknowledgement latency Y (in milliseconds) of a Caspian currency-trading platform has approximately a gamma distribution with mean 14.1 ms and variance 66.27 ms^2 .

Give non-integer answers to at least four significant figures.

- (a) $\alpha =$ _____
- (b) $\beta =$ _____
- (c) Tchebysheff's theorem says that at least 75% of latencies lie within two standard deviations of the mean. Give the upper end of that interval, $\mu + 2\sigma =$ _____ ms
- (d) Your α is a whole number, so the Poisson-sum formula of Exercise 4.99 applies. Find the exact probability $P(Y > \mu + 2\sigma) =$ _____
- (e) The service-level agreement promises acknowledgement within 18.33 ms. Find $P(Y > 18.33) =$ _____

Problem 14

seed 20260925

The waiting time Y (in days, measured from 00:00 on 1 December) until the next claim on a Baku travel-insurance book is exponential with mean $\beta = 3.2$ days. The claims system records only the **day** on which the claim is lodged: $X = k$ if $k - 1 \leq Y < k$, for $k = 1, 2, \dots$ (so $X = 1$ means 1 December).

Give every answer to at least four significant figures.

- (a) $P(X = 1) =$ _____
- (b) $P(X = 5) =$ _____
- (c) X turns out to be geometric. Its mean is $E(X) =$ _____ days

(d) The probability that no claim is lodged on any of the first 5 days: $P(X > 5) =$ _____

seed 20260925

Problem 15

An operational-risk team scores each loss event with a severity index Y whose density is

$$f(y) = \begin{cases} ky^2 e^{-y/2}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give non-integer answers to at least four significant figures.

(a) Y has a gamma distribution. Its shape parameter is $\alpha =$ _____

(b) The constant that makes f a density: $k =$ _____

(You may enter an arithmetic expression such as $1/(3^4 * 6)$.)

(c) Does Y have a chi-square distribution?

☐ Yes: it is a chi-square density with 6 degrees of freedom

☐ No: it is a gamma density, but not a chi-square density

(d) $E(Y) =$ _____

(e) The standard deviation of Y : _____

(f) Events with index above 10.8 are escalated to the board. $P(Y > 10.8) =$ _____

seed 20260925

Problem 16

A supplier sells enterprise drives to a Baku data centre. A drive's lifetime Y (in months) is exponential with mean $\beta = 48$ months. If a drive fails within the warranty period, the supplier replaces it at a cost of 290 AZN; otherwise the supplier pays nothing.

Give every answer to at least four significant figures.

(a) With a 30-month warranty, the probability that a drive fails under warranty: _____

(b) The supplier's expected warranty cost per drive with a 30-month warranty: _____ AZN

(c) The supplier budgets 36 AZN of expected warranty cost per drive. The longest warranty period w^* whose expected cost stays within budget: $w^* =$ _____ months

Problem 17

seed 20260925

Large property claims reach a Baku insurer so that the number N_t of large claims in t days is Poisson with mean λt , $\lambda = 0.80$ per day, with independent counts in non-overlapping periods. A reinsurance treaty is triggered by the 3th large claim of the month. Let T be the time (in days, from the start of the month) until the treaty is triggered.

(a) Explain to yourself why $\{T > t\} = \{N_t \leq 2\}$. Deduce that T has a gamma distribution with $\alpha = 3$ and $\beta = 1/\lambda$. Then $E(T) =$ _____ days

(b) $V(T) =$ _____

(c) The probability that the treaty has **not** been triggered after 5.0 days: $P(T > 5.0) =$ _____

(d) The probability of at least 3 large claims in the first 5.0 days: _____

(e) The probability that T falls within one standard deviation of its mean, $P(\mu - \sigma < T < \mu + \sigma) =$

Give every answer to at least four significant figures.

Problem 18

seed 20260925

The monthly unplanned downtime Y (in hours) of a gas turbine at an Absheron power plant has a gamma distribution with $\alpha = 2.5$ and $\beta = 2.8$. Lost sales plus contractual penalties give a monthly loss (in AZN) of

$$L = 250Y + 55Y^2.$$

Carry full precision in intermediate steps; give every answer to at least four significant figures.

(a) $E(Y^2) =$ _____

(b) $E(L) =$ _____ AZN

(c) $E(Y^3) =$ _____

(d) $V(L) =$ _____ AZN^2

Problem 19

seed 20260925

A Baku data centre leases a GPU server to a fintech client for a contract of 48 months. The client pays 220 AZN per month while the server runs; rent stops if the server fails. If the server fails before the contract ends, the data centre also pays the client a penalty of 2700 AZN. The server's lifetime Y (in months) is exponential with mean $\beta = 21$ months, and a failed server is not replaced.

The number of months of rent received is $M = \min(Y, 48)$.

Give every answer to at least four significant figures.

(a) The probability that the server survives the whole contract: $P(Y > 48) =$ _____

(b) $E(M) = E[\min(Y, 48)] =$ _____ months

(c) The data centre's expected net income from the contract (rent minus expected penalty): _____
AZN

(d) The penalty K^* at which the expected net income would be exactly zero: $K^* =$ _____ AZN

Problem 20

seed 20260925

A Baku data centre must buy a power-distribution unit that has to run for the next 18 months without failing. If the unit fails during those 18 months, the outage and emergency replacement cost 4500 AZN. Two suppliers offer units:

- Supplier A: price 830 AZN; lifetime exponential with mean 32 months.
- Supplier B: price 590 AZN; lifetime gamma with $\alpha = 2$ and $\beta = 38$, so mean 76 months.

Let Y_A, Y_B be the lifetimes. Give every answer to at least four significant figures.

(a) $P(Y_A \leq 18) =$ _____

(b) $P(Y_B \leq 18) =$ _____

(c) The expected total cost (price plus expected failure cost) with Supplier A: AZN

(d) The expected total cost with Supplier B: AZN

(e) Which supplier minimises expected total cost?

- ☐ Supplier A
- ☐ Supplier B

