Week 10, Problem Set 2 - worked solutions

Wackerly 4.6 - The Gamma, Exponential and Chi-Square Distributions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku brokerage records the time Y (in minutes) between consecutive large trades (single orders above 500,000 AZN) in its order book. Over a normal trading session Y has an exponential distribution (Definition 4.11) with $\beta = 12$ minutes.

Give every probability to at least four significant figures.

(a) The probability that the next gap exceeds 23 minutes: $P(Y > 23) =$ ____________

(b) The probability that the next gap is at most 14 minutes: $P(Y \le 14) =$ ____________

(c) $P(14 < Y \le 23) =$ ____________

(d) The standard deviation of the gap, in minutes: $\sigma =$ ____________

By Definition 4.11, $f(y) = \frac{1}{\beta} e^{-y/\beta}$ for $y \ge 0$. For $y \ge 0$,

$$P(Y > y) = \int_y^\infty \frac{1}{\beta} e^{-t/\beta}\, dt = \Big[-e^{-t/\beta} \Big]_y^\infty = e^{-y/\beta}, \qquad F(y) = 1 - e^{-y/\beta}.$$

(a) $P(Y > 23) = e^{-23/12} = e^{-1.9167} = 0.14710$.

(b) $P(Y \le 14) = 1 - e^{-14/12} = 1 - e^{-1.1667} = 1 - 0.31140 = 0.68860$.

(c) The interval probability is a difference of survival probabilities:

$$P(14 < Y \le 23) = P(Y > 14) - P(Y > 23) = 0.31140 - 0.14710 = 0.16431.$$

(For a continuous variable it does not matter whether the endpoints are included.)

(d) By Theorem 4.10, $V(Y) = \beta^2$, so $\sigma = \beta = 12$ minutes. The exponential is the one member of the gamma family whose standard deviation equals its mean: gaps are as variable as they are long, which is why a quiet spell of twice the average gap is not unusual.

Problem 2

seed 20260925

A motor insurer in Baku models the waiting time Y (in days) between successive claims on one fleet policy as exponential with unknown mean β. Its claims history shows that the policy goes more than 6 days without a claim with probability

$$P(Y > 6) = 0.32.$$

Give every answer to at least four significant figures.

(a) $\beta = E(Y) =$ ____________ days

(b) $P(Y \le 6.6) =$ ____________

(c) $V(Y) =$ ____________ days^2

For an exponential variable (Definition 4.11), $P(Y > y) = \int_y^\infty \beta^{-1} e^{-t/\beta}\, dt = e^{-y/\beta}$.

(a) Set $e^{-6/\beta} = 0.32$ and take logarithms: $-6/\beta = \ln 0.32 = -1.13943$, so

$$\beta = \frac{6}{-\ln 0.32} = \frac{6}{1.13943} = 5.26577 \text{ days}.$$

(b) $P(Y \leq 6.6) = 1 - e^{-6.6/\beta} = 1 - e^{-1.25338} = 0.71446$.

(c) By Theorem 4.10, $V(Y) = \beta^2 = (5.26577)^2 = 27.7283$.

One survival probability pins down the whole distribution, because the exponential family has a single parameter. That is also its weakness as a model: the insurer cannot fit a mean and a spread separately.

Problem 3

seed 20260925

The service time Y (in minutes) of a call to a bank's customer-support line has a gamma distribution (Definition 4.9) with $\alpha = 2.0$ and $\beta = 3.8$. Each call costs the bank a fixed 1.90 AZN for the telephony platform plus 0.35 AZN per minute of agent time, so the cost of a call is $C = 1.90 + 0.35\,Y$ AZN.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________ minutes

(b) $V(Y) =$ ______________

(c) $\sigma_Y =$ ______________ minutes

(d) $E(C) =$ ______________ AZN

(e) $V(C) =$ ______________

(a)-(b) Theorem 4.8 gives the moments of a gamma variable directly:

$$E(Y) = \alpha\beta = 2.0(3.8) = 7.6000, \qquad V(Y) = \alpha\beta^2 = 2.0(3.8)^2 = 28.8800.$$

(The proof integrates $y\,f(y)$ and recognises $\int_0^\infty y^\alpha e^{-y/\beta}\,dy = \beta^{\alpha+1}\Gamma(\alpha+1)$, then uses $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.)

(c) $\sigma_Y = \sqrt{28.8800} = 5.3740$ minutes.

(d) Expectation is linear (Theorem 4.5): $E(C) = 1.90 + 0.35\,E(Y) = 1.90 + 0.35(7.6000) = 4.5600$ AZN.

(e) The fixed charge does not vary, and the per-minute rate scales the variance by its square:
$V(C) = 0.35^2\,V(Y) = 0.35^2(28.8800) = 3.53780$.

Note that α alone sets the shape: the coefficient of variation $\sigma_Y/E(Y) = 1/\sqrt{\alpha}$ does not depend on β.

Problem 4

seed 20260925

A bank's model-validation team backtests its Value-at-Risk model each quarter. When the model is correctly specified, the backtest statistic Y has a chi-square distribution with $\nu = 10$ degrees of freedom.

(a) $E(Y) =$ ______________

(b) $V(Y) =$ ______________

(c) By Definition 4.10, Y is a gamma random variable with $\beta = 2$ and $\alpha =$ ______________

(d) For its dashboard the team reports the rescaled score $W = 4.65\,Y$. W is again gamma distributed with the same α. Find $E(W) =$

____

(e) $V(W) =$ ______________

Give non-integer answers to at least four significant figures.

(a)-(b) Definition 4.10 makes a chi-square variable with ν degrees of freedom a gamma variable with $\alpha = \nu/2$ and $\beta = 2$, so Theorem 4.8 gives Theorem 4.9:

$$E(Y) = \alpha\beta = \frac{\nu}{2} \cdot 2 = \nu = 10, \qquad V(Y) = \alpha\beta^2 = \frac{\nu}{2} \cdot 4 = 2\nu = 20.$$

(c) $\alpha = \nu/2 = 10/2 = 5.0$.

(d) Multiplying a gamma variable by a positive constant keeps the shape α and multiplies the scale β by that constant (this is why β is called the scale parameter in Section 4.6). So W is gamma with $\alpha = 5.0$ and $\beta = 2(4.65) = 9.30$, and

$$E(W) = \alpha\beta = 5.0(9.30) = 46.5000.$$

This equals $4.65\,E(Y)$, as linearity of expectation requires.

(e) $V(W) = \alpha\beta^2 = 5.0(9.30)^2 = 432.4500$, which equals $(4.65)^2V(Y)$. Note that W is **not** chi-square unless the constant is 1: its scale parameter is no longer 2.

Problem 5

seed 20260925

A data centre in Sumgait buys solid-state drives whose lifetime Y (in months) is exponentially distributed with mean $\beta = 37$ months.

Give every answer to at least four significant figures.

(a) The median lifetime $\phi_{0.5}$, the value with $P(Y \le \phi_{0.5}) = 0.5$: ______________ months

(b) The supplier wants a warranty period w short enough that only 14% of drives fail before it, that is $P(Y \le w) = 0.14$. $w =$ ______________ months

(c) The probability that a drive fails within its first 18 months: $P(Y \le 18) =$ ______________

For the exponential distribution (Definition 4.11) the distribution function is $F(y) = \int_0^y \beta^{-1}e^{-t/\beta}\,dt = 1 - e^{-y/\beta}$ for $y \ge 0$. A quantile ϕ_p solves $1 - e^{-\phi_p/\beta} = p$, so

$$\phi_p = -\beta \ln(1-p).$$

(a) With $p = 0.5$: $\phi_{0.5} = \beta \ln 2 = 37(0.693147) = 25.6464$ months.

The median is only about 69% of the mean: the exponential density is skewed to the right, and a minority of very long-lived drives pull the mean up.

(b) With $p = 0.14$: $w = -37\ln(0.86) = -37(-0.15082) = 5.5804$ months.

(c) $P(Y \le 18) = 1 - e^{-18/37} = 1 - e^{-0.48649} = 0.38522$.

Problem 6

seed 20260925

Customers phoning a mobile operator's billing line wait on hold for a time Y (in minutes) that is exponentially distributed with mean $\beta = 4.0$ minutes. A customer has already been on hold for 10 minutes.

Give every probability to at least four significant figures.

(a) The probability that she is still on hold 3.5 minutes from now: $P(Y > 13.5 \mid Y > 10) =$ ______________

(b) For a customer who has just joined the queue, $P(Y > 3.5) =$ ______________

(c) For a customer who has just joined the queue, $P(Y > 13.5) =$ ______________

(d) The expected **additional** hold time of the customer who has already waited 10 minutes, $E(Y - 10 \mid Y > 10) =$ ______________ minutes

For an exponential variable, $P(Y > y) = e^{-y/\beta}$.

(a) The event $\{Y > 13.5\}$ is contained in $\{Y > 10\}$, so their intersection is $\{Y > 13.5\}$ and, as in Example 4.10,

$$P(Y > 13.5 \mid Y > 10) = \frac{P(Y > 13.5)}{P(Y > 10)} = \frac{e^{-13.5/4.0}}{e^{-10/4.0}} = \frac{0.03422}{0.08208} = e^{-3.5/4.0} = 0.41686.$$

(b) $P(Y > 3.5) = e^{-3.5/4.0} = 0.41686$, the same number as (a). This is the memoryless property: the 10 minutes already spent on hold tell us nothing about how much longer the wait will be.

(c) $P(Y > 13.5) = e^{-13.5/4.0} = 0.03422$, which is smaller than (a). Before joining the queue, a wait of 13.5 minutes in total is less likely than it is once 10 minutes have already passed.

(d) Part (a) holds for every $b > 0$: $P(Y - 10 > b \mid Y > 10) = e^{-b/\beta}$. So given $Y > 10$, the remaining wait $Y - 10$ is again exponential with mean β, and its expected value is $\beta = 4.0$ minutes, whatever 10 was.

Problem 7

seed 20260925

During the afternoon session, large trades in a Baku brokerage's order book arrive so that the number of large trades in any t minutes has a Poisson distribution with mean λt, where $\lambda = 0.15$ per minute, and counts in non-overlapping periods are independent. Let W be the waiting time (in minutes) from 14:00 until the first large trade.

Give every number to at least four significant figures.

(a) The probability of no large trade in the first 10 minutes: __________

(b) The event $\{W > w\}$ is the event "no large trade in the first w minutes". Use this to find the distribution function of W for $w \geq 0$: $F(w) = P(W \leq w) =$ __________

(Enter a function of w; write the exponential as e^(...) or exp(...).)

(c) W is exponential (Definition 4.11). Its mean $\beta = E(W) =$ __________ minutes

(d) The median waiting time: __________ minutes

(e) $P(W \leq 2.0) =$ __________

(f) The probability of at least two large trades in the first 10 minutes: __________

(a) The count N in 10 minutes is Poisson with mean $\lambda t = 0.15(10) = 1.5000$, so $P(N = 0) = e^{-1.5000} = 0.22313$.

(b) The first trade arrives after w exactly when there is no trade in $[0, w]$, so $P(W > w) = P(N_w = 0) = e^{-\lambda w}$ and

$$F(w) = 1 - e^{-\lambda w} = 1 - e^{-0.15w}, \qquad w \geq 0.$$

(c) Differentiating, $f(w) = \lambda e^{-\lambda w} = \frac{1}{\beta} e^{-w/\beta}$ with $\beta = 1/\lambda$. That is Definition 4.11, so by Theorem 4.10

$$E(W) = \beta = \frac{1}{\lambda} = \frac{1}{0.15} = 6.6667 \text{ minutes.}$$

A rate of 0.15 trades per minute and a mean gap of 6.6667 minutes are the same statement.

(d) Solve $1 - e^{-\lambda m} = 0.5$: $m = \ln 2/\lambda = 0.693147/0.15 = 4.6210$ minutes.

(e) $P(W \leq 2.0) = 1 - e^{-0.15(2.0)} = 1 - e^{-0.3000} = 0.25918$.

(f) Back to the Poisson count with mean 1.5000:

$$P(N \geq 2) = 1 - P(N = 0) - P(N = 1) = 1 - e^{-1.5000}(1 + 1.5000) = 1 - 0.22313 - 0.33470 = 0.44217.$$

The same arrivals can be described by counts (Poisson, discrete) or by gaps (exponential, continuous); part (b) is the bridge between them.

Problem 8

seed 20260925

The time Y (in working days) that a Baku bank takes to process a mortgage application, from submission to a credit decision, has a gamma distribution with $\alpha = 2$ and $\beta = 6.0$.

Because α is a whole number, gamma probabilities can be written as sums of Poisson probabilities (Exercise 4.99): for $\lambda > 0$,

$$\frac{1}{\Gamma(\alpha)} \int_{\lambda}^{\infty} u^{\alpha-1} e^{-u}\, du = \sum_{x=0}^{\alpha-1} \frac{\lambda^x e^{-\lambda}}{x!}.$$

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________ days

(b) The probability that a decision takes more than 14.4 working days: $P(Y > 14.4) =$ ______________

(c) The probability that a decision arrives within 21.6 working days: $P(Y \le 21.6) =$ ______________

(a) Theorem 4.8: $E(Y) = \alpha\beta = 2(6.0) = 12.00$ days.

(b) Start from Definition 4.9 and substitute $u = y/\beta$, $dy = \beta\, du$:

$$P(Y > c) = \int_c^{\infty} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dy = \frac{1}{\Gamma(\alpha)} \int_{c/\beta}^{\infty} u^{\alpha-1} e^{-u}\, du = \sum_{x=0}^{\alpha-1} \frac{(c/\beta)^x e^{-c/\beta}}{x!}.$$

So $P(Y > c)$ is the probability that a Poisson variable with mean $\lambda = c/\beta$ is at most $\alpha - 1 = 1$. Here $\lambda = 14.4/6.0 = 2.4000$, and the 2 Poisson terms $x = 0, \ldots, 1$ are

$$P(Y > 14.4) = 0.09072 + 0.21772 = 0.30844.$$

(c) Now $\lambda = 21.6/6.0 = 3.6000$:

$$P(Y > 21.6) = 0.02732 + 0.09837 = 0.12569,$$

so $P(Y \le 21.6) = 1 - 0.12569 = 0.87431$.

For a non-integer α no such finite sum exists and the integral has to be done numerically (for example with pgamma in R).

Problem 9

seed 20260925

When a cooling failure takes a server hall offline at a Baku data centre, the outage lasts Y hours, exponentially distributed with mean $\beta = 1.2$ hours. Penalties to clients grow faster than the outage itself, and the total cost (in thousands of AZN) is

$$C = 10 + 15\,Y + 5.5\,Y^2.$$

Give every answer to at least four significant figures.

(a) $E(Y^2) =$ ______________

(b) $E(C) =$ ______________ thousand AZN

(c) $V(C) =$ ______________

(Hint: you will need $E(Y^3)$ and $E(Y^4)$.)

Moments of the exponential. With Definition 4.11 and the gamma integral $\int_0^\infty y^k e^{-y/\beta}\, dy = \beta^{k+1}\Gamma(k+1)$,

$$E(Y^k) = \frac{1}{\beta} \int_0^{\infty} y^k e^{-y/\beta}\, dy = \beta^k\, \Gamma(k+1) = k!\, \beta^k.$$

So $E(Y) = \beta = 1.2$, $E(Y^2) = 2\beta^2$, $E(Y^3) = 6\beta^3 = 10.3680$, $E(Y^4) = 24\beta^4 = 49.7664$.

(a) $E(Y^2) = 2(1.2)^2 = 2.8800$. (Equivalently $V(Y) + [E(Y)]^2 = \beta^2 + \beta^2$ by Theorem 4.10.)

(b) By linearity (Theorem 4.5),

$$E(C) = 10 + 15(1.2) + 5.5(2.8800) = 43.8400.$$

(c) The constant drops out, and $V(aY + bY^2) = a^2V(Y) + b^2V(Y^2) + 2ab\,\mathrm{Cov}(Y, Y^2)$, with

$$V(Y) = \beta^2 = 1.4400, \quad V(Y^2) = E(Y^4) - [E(Y^2)]^2 = 20\beta^4 = 41.4720, \quad \mathrm{Cov}(Y, Y^2) = E(Y^3) - E(Y)E(Y^2) = 4\beta^3 = 6.9120.$$

Therefore

$$V(C) = 15^2(1.4400) + 5.5^2(41.4720) + 2(15)(5.5)(6.9120) = 2719.0080.$$

Notice that $E(C) \neq 10 + 15\,E(Y) + 5.5\,[E(Y)]^2$: plugging the mean outage into a convex cost understates the expected cost, because long outages are penalised quadratically.

Problem 10

seed 20260925

A colocation provider models the lifetime Y (in months) of the power supply in a rack server as exponential with mean $\beta = 32$ months. An auditor finds a server whose power supply has already worked for 45 months.

Give every answer to at least four significant figures.

(a) The probability that this power supply is still working 18 months from now: $P(Y > 63 \mid Y > 45) =$ ____________

(b) For comparison, the probability that a **new** power supply lasts 63 months: $P(Y > 63) =$ ____________

(c) The expected total lifetime of the audited power supply, $E(Y \mid Y > 45) =$ ____________ months

(d) The median total lifetime of the audited power supply, the value m with $P(Y > m \mid Y > 45) = 0.5$: $m =$ ____________ months

For an exponential variable, $P(Y > y) = e^{-y/\beta}$. For any $s \geq 0$, as in Example 4.10,

$$P(Y > 45 + s \mid Y > 45) = \frac{e^{-(45+s)/\beta}}{e^{-45/\beta}} = e^{-s/\beta}.$$

So, given $Y > 45$, the residual life $R = Y - 45$ is again exponential with mean $\beta = 32$.

(a) With $s = 18$: $e^{-18/32} = e^{-0.56250} = 0.56978$.

(b) $P(Y > 63) = e^{-63/32} = e^{-1.96875} = 0.13963$, smaller than (a) because the new unit must also survive the first 45 months.

(c) $E(Y \mid Y > 45) = 45 + E(R) = 45 + 32 = 77$ months. An old unit is expected to last longer **in total** than a new one (whose mean is 32), simply because it has already survived; but its expected **remaining** life is exactly that of a new unit.

(d) The median of R is $\beta \ln 2 = 22.1807$ (solve $e^{-s/\beta} = 0.5$), so $m = 45 + 22.1807 = 67.1807$ months.

If power supplies wear out, the exponential model is wrong for them, and the auditor should expect **less** remaining life for older units; the memoryless property is a strong assumption, not a harmless convenience.

Problem 11

seed 20260925

A bank's backup system stores its transaction archive on an array of 3 drives. Each drive sits on its own controller, with its own power feed, in a different rack, so the drive lifetimes are independent; each lifetime is exponential with mean $\beta = 6.5$ years. The array keeps the archive readable as long as at least 1 of the 3 drives work (it tolerates 2 failed drive(s)). No drives are replaced.

Give every answer to at least four significant figures.

(a) The probability that one particular drive is still working after 2.75 years: ____________

(b) The probability that the array still keeps the archive readable after 2.75 years: ______________

(c) The expected number of drives still working after 2.75 years: ______________

(a) For one exponential lifetime Y_i, $P(Y_i > 2.75) = \int_{2.75}^{\infty} \beta^{-1} e^{-y/\beta}\, dy = e^{-2.75/6.5} = e^{-0.42308} = 0.65503$.

(b) Let X be the number of drives still working at 2.75 years. Each drive independently works with probability $p = 0.65503$, so X is binomial with $n = 3$ and $p = 0.65503$ (Definition 3.7). The array works when $X \geq 1$:

$$P(X \geq 1) = \sum_{j=1}^{3} \binom{3}{j} p^j (1-p)^{3-j} = 0.23386 + 0.44404 + 0.28105 = 0.95895,$$

with $1 - p = 0.34497$.

(c) $E(X) = np = 3(0.65503) = 1.9651$ (Theorem 3.7).

The continuous model supplies one number, the survival probability of a single drive, and the discrete binomial model does the rest. The independence assumption is doing real work: drives on a shared power feed could all fail together, and the binomial count would then understate the risk.

Problem 12

seed 20260925

At a Baku insurer's claims call centre, the length Y (in minutes) of a call is exponential with mean $\beta = 4.0$ minutes. A call longer than 19 minutes is escalated to a supervisor. Consider the next 11 calls, whose lengths are independent.

Give every answer to at least four significant figures.

(a) The probability that a single call is escalated: $P(Y > 19) =$ ______________

(b) The probability that at least one of the next 11 calls is escalated: ______________

(c) The probability that exactly two of the next 11 calls are escalated: ______________

(d) The expected number of escalations among the next 11 calls: ______________

(a) $P(Y > 19) = e^{-19/4.0} = e^{-4.75000} = 0.008652$.

Let X be the number of escalated calls among the next 11. Each call independently exceeds 19 minutes with probability $p = 0.008652$, so X is binomial with $n = 11$, $p = 0.008652$.

(b) $P(X \geq 1) = 1 - P(X = 0) = 1 - (1-p)^{11} = 1 - (0.991348)^{11} = 1 - 0.908843 = 0.091157$.

(c) $P(X = 2) = \binom{11}{2} p^2 (1-p)^9 = 55(0.008652)^2 (0.991348)^9 = 0.003807$.

(d) $E(X) = np = 11(0.008652) = 0.09517$.

"At least one" is always easiest through the complement: none of the 11 calls is long.

Problem 13

seed 20260925

The order-acknowledgement latency Y (in milliseconds) of a Caspian currency-trading platform has approximately a gamma distribution with mean 14.1 ms and variance 66.27 ms^2.

Give non-integer answers to at least four significant figures.

(a) $\alpha =$ ______________

(b) $\beta =$ ______________

(c) Tchebysheff's theorem says that at least 75% of latencies lie within two standard deviations of the mean. Give the upper end of that interval, $\mu + 2\sigma =$ ______________ ms

(d) Your α is a whole number, so the Poisson-sum formula of Exercise 4.99 applies. Find the exact probability $P(Y > \mu + 2\sigma) =$

____

(e) The service-level agreement promises acknowledgement within 18.33 ms. Find $P(Y > 18.33) =$ ______________

(a)-(b) By Theorem 4.8, $\mu = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$. Dividing, $\beta = \sigma^2/\mu$; then $\alpha = \mu/\beta = \mu^2/\sigma^2$:

$$\beta = \frac{66.27}{14.1} = 4.7, \qquad \alpha = \frac{(14.1)^2}{66.27} = 3.$$

(c) $\sigma = \sqrt{66.27} = 8.1406$, so the interval is $14.1 \pm 2(8.1406) = (-2.1813, 30.3813)$. (Its lower end is below zero when $\alpha < 4$, and latencies cannot be negative.)

(d) From Exercise 4.99, substituting $u = y/\beta$, $P(Y > c) = \sum_{x=0}^{\alpha-1} \lambda^x e^{-\lambda}/x!$ with $\lambda = c/\beta$. Here $\lambda = 30.3813/4.7 = 6.4641$ (which is $\alpha + 2\sqrt{\alpha}$), and

$$P(Y > 30.3813) = 0.00156 + 0.01007 + 0.03256 = 0.04419.$$

Tchebysheff only promises that **both** tails together hold at most 0.25; the exact upper tail is far smaller. The bound holds for every distribution with this mean and variance, which is why it is so conservative for this one.

(e) $\lambda = 18.33/4.7 = 3.9000$, and

$$P(Y > 18.33) = 0.02024 + 0.07894 + 0.15394 = 0.25313.$$

Problem 14

seed 20260925

The waiting time Y (in days, measured from 00:00 on 1 December) until the next claim on a Baku travel-insurance book is exponential with mean $\beta = 3.2$ days. The claims system records only the **day** on which the claim is lodged: $X = k$ if $k - 1 \le Y < k$, for $k = 1, 2, \ldots$ (so $X = 1$ means 1 December).

Give every answer to at least four significant figures.

(a) $P(X = 1) =$ ______________

(b) $P(X = 5) =$ ______________

(c) X turns out to be geometric. Its mean is $E(X) =$ ______________ days

(d) The probability that no claim is lodged on any of the first 5 days: $P(X > 5) =$ ______________

Using $P(Y > y) = e^{-y/\beta}$, for each $k = 1, 2, \ldots$

$$P(X = k) = P(k - 1 \le Y < k) = e^{-(k-1)/\beta} - e^{-k/\beta} = \left(e^{-1/\beta}\right)^{k-1}\left(1 - e^{-1/\beta}\right).$$

That is the geometric probability function (Definition 3.8) with success probability $p = 1 - e^{-1/\beta}$; here $e^{-1/\beta} = e^{-1/3.2} = 0.731616$.

(a) $P(X = 1) = p = 1 - 0.731616 = 0.268384$.

(b) $P(X = 5) = (0.731616)^4(0.268384) = 0.076893$.

(c) By Theorem 3.8, $E(X) = 1/p = 1/0.268384 = 3.72600$ days. This is about $\beta + 1/2$, as one would expect from rounding a continuous time up to the end of its day.

(d) $X > 5$ exactly when $Y \ge 5$, so $P(X > 5) = e^{-5/3.2} = (1 - p)^5 = 0.209611$.

Both distributions are memoryless (Example 4.10 for the exponential; the geometric has the discrete analogue noted after it), which is why discretising one produces the other.

Problem 15

seed 20260925

An operational-risk team scores each loss event with a severity index Y whose density is

$$f(y) = \begin{cases} k\,y^2 e^{-y/2}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give non-integer answers to at least four significant figures.

(a) Y has a gamma distribution. Its shape parameter is $\alpha =$ ______________

(b) The constant that makes f a density: $k =$ ______________

(You may enter an arithmetic expression such as 1/(3^4 * 6).)

(c) Does Y have a chi-square distribution?

- ○ Yes: it is a chi-square density with 6 degrees of freedom
- ○ No: it is a gamma density, but not a chi-square density

(d) $E(Y) =$ ______________

(e) The standard deviation of Y: ______________

(f) Events with index above 10.8 are escalated to the board. $P(Y > 10.8) =$ ______________

(a) Compare with Definition 4.9, $f(y) = y^{\alpha-1}e^{-y/\beta}/(\beta^\alpha\Gamma(\alpha))$: the power $y^2 = y^{\alpha-1}$ gives $\alpha = 3$, and the exponent gives $\beta = 2$.

(b) k must equal $1/(\beta^\alpha\Gamma(\alpha))$, and $\Gamma(3) = 2! = 2$, so

$$k = \frac{1}{(2)^3 \cdot 2} = \frac{1}{16.0000} = 0.0625000.$$

(c) By Definition 4.10, a chi-square variable is a gamma variable with $\beta = 2$ (and $\alpha = \nu/2$). Here $\beta = 2$, so the answer is: Yes: it is a chi-square density with 6 degrees of freedom. (Whenever $\beta = 2$ and 2α is a positive integer, the degrees of freedom are $\nu = 2\alpha$.)

(d) Theorem 4.8: $E(Y) = \alpha\beta = 3(2) = 6$.

(e) $V(Y) = \alpha\beta^2$, so $\sigma = \sqrt{\alpha}\,\beta = \sqrt{3}(2) = 3.4641$.

(f) α is an integer, so by Exercise 4.99 (substitute $u = y/\beta$) $P(Y > c) = \sum_{x=0}^{\alpha-1} \lambda^x e^{-\lambda}/x!$ with $\lambda = c/\beta = 10.8/2 = 5.4000$:

$$P(Y > 10.8) = 0.00452 + 0.02439 + 0.06585 = 0.09476.$$

Problem 16

seed 20260925

A supplier sells enterprise drives to a Baku data centre. A drive's lifetime Y (in months) is exponential with mean $\beta = 48$ months. If a drive fails within the warranty period, the supplier replaces it at a cost of 290 AZN; otherwise the supplier pays nothing.

Give every answer to at least four significant figures.

(a) With a 30-month warranty, the probability that a drive fails under warranty: ______________

(b) The supplier's expected warranty cost per drive with a 30-month warranty: ______________ AZN

(c) The supplier budgets 36 AZN of expected warranty cost per drive. The longest warranty period w^* whose expected cost stays within budget: $w^* =$ ______________ months

(a) $P(Y \le 30) = 1 - e^{-30/48} = 1 - e^{-0.62500} = 0.46474$.

(b) The cost is the random variable $C = 290$ if $Y \le 30$ and $C = 0$ otherwise, so

$$E(C) = 290\,P(Y \le 30) + 0 \cdot P(Y > 30) = 290(0.46474) = 134.7742 \text{ AZN}.$$

(c) For a warranty of length w, $E(C) = 290(1 - e^{-w/\beta})$, which increases with w. The largest allowed w solves $290(1 - e^{-w/48}) = 36$:

$$e^{-w^*/48} = 1 - \frac{36}{290} = 0.87586 \quad\Longrightarrow\quad w^* = -48\ln(0.87586) = -48(-0.13255) = 6.3622 \text{ months}.$$

This is the quantile $\phi_p = -\beta\ln(1-p)$ of the lifetime at $p = 36/290$: the budget fixes the fraction of drives the supplier can afford to replace.

Problem 17

seed 20260925

Large property claims reach a Baku insurer so that the number N_t of large claims in t days is Poisson with mean λt, $\lambda = 0.80$ per day, with independent counts in non-overlapping periods. A reinsurance treaty is triggered by the 3th large claim of the month. Let T be the time (in days, from the start of the month) until the treaty is triggered.

(a) Explain to yourself why $\{T > t\} = \{N_t \le 2\}$. Deduce that T has a gamma distribution with $\alpha = 3$ and $\beta = 1/\lambda$. Then $E(T) =$ ____________ days

(b) $V(T) =$ ____________

(c) The probability that the treaty has **not** been triggered after 5.0 days: $P(T > 5.0) =$ ____________

(d) The probability of at least 3 large claims in the first 5.0 days: ____________

(e) The probability that T falls within one standard deviation of its mean, $P(\mu - \sigma < T < \mu + \sigma) =$ ____________

Give every answer to at least four significant figures.

(a) The 3th claim arrives after day t exactly when at most 2 claims have arrived by day t. Hence

$$P(T > t) = P(N_t \le 2) = \sum_{x=0}^{2} \frac{(\lambda t)^x e^{-\lambda t}}{x!}.$$

Differentiating $F(t) = 1 - P(T > t)$, the sum telescopes:

$$f(t) = \sum_{x=0}^{2} \left[\frac{\lambda(\lambda t)^x e^{-\lambda t}}{x!} - \frac{\lambda(\lambda t)^{x-1} e^{-\lambda t}}{(x-1)!}\right] = \frac{\lambda^3 t^2 e^{-\lambda t}}{2!},$$

(the second term is absent for $x = 0$), which is Definition 4.9 with $\alpha = 3$, $\beta = 1/\lambda = 1.25000$; this is the identity of Exercise 4.99 read backwards. By Theorem 4.8, $E(T) = \alpha\beta = 3/0.80 = 3.7500$ days.

(b) $V(T) = \alpha\beta^2 = 3/0.80^2 = 4.6875$, so $\sigma = 2.1651$.

(c) $\lambda t = 0.80(5.0) = 4.0000$, and

$$P(T > 5.0) = 0.01832 + 0.07326 + 0.14653 = 0.23810.$$

(d) $P(N_{5.0} \ge 3) = 1 - P(N_{5.0} \le 2) = P(T \le 5.0) = 1 - 0.23810 = 0.76190$.

(e) The interval is $(1.5849, 5.9151)$. Using the same Poisson sum at $\lambda(\mu \mp \sigma) = 3 \mp \sqrt{3}$, that is at 1.2679 and 4.7321,

$$P(T > \mu - \sigma) = 0.86443, \qquad P(T > \mu + \sigma) = 0.14911,$$

so $P(\mu - \sigma < T < \mu + \sigma) = 0.86443 - 0.14911 = 0.71532$. This depends on k but not on λ: changing the claim rate only rescales time.

Problem 18

seed 20260925

The monthly unplanned downtime Y (in hours) of a gas turbine at an Absheron power plant has a gamma distribution with $\alpha = 2.5$ and $\beta = 2.8$. Lost sales plus contractual penalties give a monthly loss (in AZN) of

$$L = 250\,Y + 55\,Y^2.$$

Carry full precision in intermediate steps; give every answer to at least four significant figures.

(a) $E(Y^2) =$ ______________

(b) $E(L) =$ ______________ AZN

(c) $E(Y^3) =$ ______________

(d) $V(L) =$ ______________ AZN^2

Moments of the gamma (Exercise 4.111). For $k > -\alpha$,

$$E(Y^k) = \int_0^\infty y^k \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}\,dy = \frac{\beta^{\alpha+k}\Gamma(\alpha+k)}{\beta^\alpha\Gamma(\alpha)} = \beta^k\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)},$$

because the integrand is an unnormalised gamma density with shape $\alpha + k$. Repeated use of $\Gamma(a+1) = a\Gamma(a)$ gives $E(Y^k) = \alpha(\alpha+1)\cdots(\alpha+k-1)\beta^k$:

$$E(Y) = 2.5(2.8) = 7.0000, \qquad E(Y^2) = 2.5(3.5)(2.8)^2 = 68.6000,$$

$$E(Y^3) = 2.5(3.5)(4.5)(2.8)^3 = 864.3600, \qquad E(Y^4) = 2.5(3.5)(4.5)(5.5)(2.8)^4 = 13311.1440.$$

(a) $E(Y^2) = 68.6000$. Check: $V(Y) + [E(Y)]^2 = \alpha\beta^2 + \alpha^2\beta^2$, the same.

(b) $E(L) = 250\,E(Y) + 55\,E(Y^2) = 250(7.0000) + 55(68.6000) = 5523.0000$ AZN.

(c) $E(Y^3) = 864.3600$, from the formula above.

(d) $L^2 = 250^2Y^2 + 2(250)(55)Y^3 + 55^2Y^4$, so

$$E(L^2) = 250^2(68.6000) + 2(250)(55)(864.3600) + 55^2(13311.1440) = 68323610.60,$$

and $V(L) = E(L^2) - [E(L)]^2 = 68323610.60 - (5523.0000)^2 = 37820081.60$.

The subtraction in (d) cancels many leading digits, which is why rounding $E(L)$ early can move $V(L)$ noticeably.

Problem 19

seed 20260925

A Baku data centre leases a GPU server to a fintech client for a contract of 48 months. The client pays 220 AZN per month while the server runs; rent stops if the server fails. If the server fails before the contract ends, the data centre also pays the client a penalty of 2700 AZN. The server's lifetime Y (in months) is exponential with mean $\beta = 21$ months, and a failed server is not replaced.

The number of months of rent received is $M = \min(Y, 48)$.

Give every answer to at least four significant figures.

(a) The probability that the server survives the whole contract: $P(Y > 48) =$ ______________

(b) $E(M) = E[\min(Y, 48)] =$ ______________ months

(c) The data centre's expected net income from the contract (rent minus expected penalty): ______________ AZN

(d) The penalty K^* at which the expected net income would be exactly zero: $K^* =$ ______________ AZN

(a) $P(Y > 48) = e^{-48/21} = e^{-2.28571} = 0.10170$.

(b) $M = Y$ when $Y \le 48$ and $M = 48$ otherwise. So, writing $T = 48$,

$$E(M) = \int_0^T y\,\frac{1}{\beta}e^{-y/\beta}\,dy + T\,P(Y > T).$$

Integrating by parts, $\int_0^T \frac{y}{\beta}e^{-y/\beta}\,dy = \Big[-ye^{-y/\beta}\Big]_0^T + \int_0^T e^{-y/\beta}\,dy = -Te^{-T/\beta} + \beta(1 - e^{-T/\beta})$. Adding $Te^{-T/\beta}$, the boundary terms cancel:

$$E(M) = \beta\big(1 - e^{-T/\beta}\big) = 21(1 - 0.10170) = 21(0.89830) = 18.8643 \text{ months}.$$

(Equivalently, $E(M) = \int_0^T P(Y > y)\,dy$.)

(c) Rent is $220M$ and the penalty is 2700 times the indicator of $\{Y \le 48\}$, so by linearity

$$E(\text{net}) = 220\,E(M) - 2700\,P(Y \le 48) = 220(18.8643) - 2700(0.89830) = 4150.1396 - 2425.4062 = 1724.7333 \text{ AZN}.$$

(d) Setting $220\,E(M) - K^*\,P(Y \le 48) = 0$:

$$K^* = \frac{220(18.8643)}{0.89830} = 220\beta = 4620.0000 \text{ AZN}.$$

The simplification $K^* = r\beta$ follows from (b): $E(M)/P(Y \le T) = \beta$ whatever the contract length, another face of memorylessness.

Problem 20

seed 20260925

A Baku data centre must buy a power-distribution unit that has to run for the next 18 months without failing. If the unit fails during those 18 months, the outage and emergency replacement cost 4500 AZN. Two suppliers offer units:

- Supplier A: price 830 AZN; lifetime exponential with mean 32 months.
- Supplier B: price 590 AZN; lifetime gamma with $\alpha = 2$ and $\beta = 38$, so mean 76 months.

Let Y_A, Y_B be the lifetimes. Give every answer to at least four significant figures.

(a) $P(Y_A \le 18) =$ ______________

(b) $P(Y_B \le 18) =$ ______________

(c) The expected total cost (price plus expected failure cost) with Supplier A: ______________ AZN

(d) The expected total cost with Supplier B: ______________ AZN

(e) Which supplier minimises expected total cost?

- ○ Supplier A
- ○ Supplier B

(a) For the exponential, $P(Y_A \le 18) = 1 - e^{-18/32} = 1 - e^{-0.56250} = 0.43022$.

(b) For a gamma variable with integer $\alpha = 2$, Exercise 4.99 (after substituting $u = y/\beta$) gives a two-term Poisson sum with $\lambda = 18/38 = 0.47368$:

$$P(Y_B > 18) = e^{-\lambda}(1 + \lambda) = 0.62270(1 + 0.47368) = 0.91767,$$

so $P(Y_B \le 18) = 1 - 0.91767 = 0.08233$. (Directly: integrate $ye^{-y/\beta}/\beta^2$ by parts.)

(c) The cost with A is $830 + 4500\,I(Y_A \le 18)$, whose expectation is

$$E(C_A) = 830 + 4500(0.43022) = 2765.9773.$$

(d) $E(C_B) = 590 + 4500(0.08233) = 960.4902$.

(e) Supplier B, since its expected cost is lower.

The mean lifetimes (32 and 76 months) never entered the calculation: what matters is the price plus the probability of failing **within** the 18 months. The two shapes differ exactly there. The gamma density with $\alpha = 2$ starts at zero, so early failures are rare (near $y = 0$, $P(Y_B \leq y) \approx y^2/(2\beta^2)$), while the exponential density is highest at $y = 0$. For a short mission a gamma unit can therefore beat an exponential unit with a longer mean, and for a long mission the ranking can reverse.

Answer key

Problem	Answers
1	(a) 0.147096, (b) 0.688597, (c) 0.164307, (d) 12
2	(a) 5.26577, (b) 0.714461, (c) 27.7283
3	(a) 7.6, (b) 28.88, (c) 5.37401, (d) 4.56, (e) 3.5378
4	(a) 10, (b) 20, (c) 5, (d) 46.5, (e) 432.45
5	(a) 25.6464, (b) 5.58045, (c) 0.385217
6	(a) 0.416862, (b) 0.416862, (c) 0.0342181, (d) 4
7	(a) 0.22313, (b) 1-e^(-0.15*w), (c) 6.66667, (d) 4.62098, (e) 0.259182, (f) 0.442175
8	(a) 12, (b) 0.308441, (c) 0.874311
9	(a) 2.88, (b) 43.84, (c) 2719.01
10	(a) 0.569783, (b) 0.139631, (c) 77, (d) 67.1807
11	(a) 0.655028, (b) 0.958946, (c) 1.96508
12	(a) 0.0086517, (b) 0.0911568, (c) 0.00380716, (d) 0.0951686
13	(a) 3, (b) 4.7, (c) 30.3813, (d) 0.0441904, (e) 0.253125
14	(a) 0.268384, (b) 0.0768934, (c) 3.726, (d) 0.209611
15	(a) 3, (b) 0.0625, (c) Yes: ... of freedom, (d) 6, (e) 3.4641, (f) 0.0947579
16	(a) 0.464739, (b) 134.774, (c) 6.36224
17	(a) 3.75, (b) 4.6875, (c) 0.238103, (d) 0.761897, (e) 0.715318
18	(a) 68.6, (b) 5523, (c) 864.36, (d) 3.78201E+07
19	(a) 0.101701, (b) 18.8643, (c) 1724.73, (d) 4620
20	(a) 0.430217, (b) 0.0823311, (c) 2765.98, (d) 960.49, (e) Supplier B