Week 11, Problem Set 1 - worked solutions

Wackerly 4.7-4.8 - The Beta Distribution and Choosing a Continuous Model

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

When a corporate borrower defaults, a bank in Baku eventually recovers a fraction Y of the outstanding balance through collateral sales and restructuring. The bank's credit-risk unit models this **recovery rate** with a beta distribution with parameters $\alpha = 2$ and $\beta = 6$ (Definition 4.12), so $0 \leq Y \leq 1$.

A loan with exposure at default of 420 thousand AZN has just defaulted.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ____________

(b) $V(Y) =$ ____________

(c) The expected amount recovered on this loan, in thousand AZN: ____________

(d) The standard deviation of the amount recovered, in thousand AZN: ____________

(a), (b) Theorem 4.11 gives, with $\alpha + \beta = 8$,

$$E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{2}{8} = 0.2500,$$
$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2 \cdot 6}{8^2 \cdot 9} = \frac{12}{576} = 0.020833.$$

(c), (d) The amount recovered is $R = 420\,Y$, a constant times Y. By Theorem 4.5, $E(cY) = cE(Y)$, and then $V(cY) = E\big[c^2(Y-\mu)^2\big] = c^2V(Y)$, so the standard deviation scales by c:

$$E(R) = 420 \times 0.2500 = 105.00, \qquad \sigma_R = 420\sqrt{V(Y)} = 420 \times 0.1443 = 60.62.$$

So the bank expects to lose about 315.00 thousand AZN on this loan, but with a standard deviation of 60.62 thousand AZN around the recovery the realised loss can be quite different. A beta model is natural here because a recovery rate is a proportion confined to $[0, 1]$, which is exactly the support of Definition 4.12.

Problem 2

seed 20260925

A new mobile operator has entered a regional market in Azerbaijan. Its share Y of the region's mobile-data revenue at the end of its first year is uncertain, and a consultant models it with the density

$$f(y) = \begin{cases} k\,y(1-y)^6, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of k that makes $f(y)$ a density function. $k =$ ____________

(b) Y has a beta distribution. Give its parameters: $\alpha =$ ____________ and $\beta =$ ____________

(c) $E(Y) =$ ____________ (at least four significant figures)

(d) The region's mobile-data revenue is 320 million AZN a year. What revenue, in million AZN, does the new operator expect to capture?

____

(b) Compare with Definition 4.12: the beta density is proportional to $y^{\alpha-1}(1-y)^{\beta-1}$ on $[0, 1]$. Matching exponents, $\alpha - 1 = 1$ and $\beta - 1 = 6$, so $\alpha = 2$ and $\beta = 7$.

(a) The constant must be $1/B(\alpha,\beta)$, and $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. For a positive integer n, $\Gamma(n)=(n-1)!$, so

$$k=\frac{\Gamma(9)}{\Gamma(2)\,\Gamma(7)}=\frac{8!}{1!\,6!}=56.$$

No integration is needed: once the density is recognised as a beta density the constant is read off from the gamma function.

(c) Theorem 4.11: $E(Y)=\alpha/(\alpha+\beta)=2/9=0.2222$.

(d) The captured revenue is $320\,Y$, so its expectation is $320\times 0.2222=71.11$ million AZN.

Problem 3

seed 20260925

A plastics factory in Sumgait loses part of each month's production capacity to maintenance, repairs and power interruptions. The proportion Y of a month's capacity that is lost has density

$$f(y)=\begin{cases}3\,(1-y)^2, & 0\le y\le 1,\\ 0, & \text{elsewhere,}\end{cases}$$

which is the beta density with $\alpha=1$ and $\beta=3$.

Give every answer to at least four significant figures.

(a) The probability that no more than 0.38 of next month's capacity is lost: $P(Y\le 0.38)=$ ____________

(b) The plant manager wants a downtime allowance y_0 (as a proportion of capacity) that will be exceeded in only 11% of months, that is $P(Y>y_0)=0.11$. $y_0=$ ____________

(c) $E(Y)=$ ____________

With $\alpha=1$ the density is a polynomial and integrates directly. For $0\le y\le 1$,

$$F(y)=\int_0^y 3\,(1-t)^2\,dt=\Big[-(1-t)^3\Big]_0^y=1-(1-y)^3.$$

(a) $P(Y\le 0.38)=F(0.38)=1-0.62^3=0.7617$.

(b) $P(Y>y_0)=1-F(y_0)=(1-y_0)^3$. Setting this equal to 0.11 gives $1-y_0=0.11^{1/3}$, so

$$y_0=1-0.11^{1/3}=0.5209.$$

Budgeting for about 52.09% of capacity lost covers the downtime in all but 11% of months. This is the $(1-0.11)$ quantile of Y.

(c) Theorem 4.11 with $\alpha=1$: $E(Y)=1/(1+3)=1/4=0.2500$.

Problem 4

seed 20260925

The daily capacity utilisation Y of an oil refinery (crude processed as a fraction of the refinery's nameplate capacity) is modelled by the beta distribution with $\alpha=4$ and $\beta=1$, so

$$f(y)=\begin{cases}4\,y^3, & 0\le y\le 1,\\ 0, & \text{elsewhere.}\end{cases}$$

(a) Find the distribution function for $0\le y\le 1$: $F(y)=$ ____________ (a formula in y)

(b) The refinery's contract with an exporter is at risk on days when utilisation is at or below 93%. Find the probability that utilisation on a given day is above 93%: $P(Y>0.93)=$ ____________

(c) The median utilisation $\phi_{0.5}$, with $F(\phi_{0.5})=0.5$: ____________

(d) Nameplate capacity is 220 thousand barrels per day. Find the expected daily throughput, in thousand barrels: ____________

Give numerical answers to at least four significant figures.

(a) For $0\le y\le 1$,

$$F(y) = \int_0^y 4\,t^3\,dt = \left[t^4\right]_0^y = y^4,$$

with $F(y) = 0$ for $y < 0$ and $F(y) = 1$ for $y > 1$.

(b) $P(Y > 0.93) = 1 - F(0.93) = 1 - 0.93^4 = 1 - 0.7481 = 0.2519$.

(c) Solve $\phi^4 = 0.5$: $\phi_{0.5} = 0.5^{1/4} = 0.8409$.

(d) Theorem 4.11: $E(Y) = \alpha/(\alpha + \beta) = 4/5 = 0.8000$, and throughput is $220\,Y$, so expected throughput is $220 \times 0.8000 = 176.00$ thousand barrels per day.

The median (0.8409) exceeds the mean (0.8000): with $\alpha > \beta$ the density piles up near full capacity and has a long left tail of bad days, the left skew explored in Exercise 4.117.

Problem 5

seed 20260925

A filling station on the Baku-Quba highway has storage tanks holding 35 thousand litres of petrol, refilled to capacity every Monday. The proportion Y of the stock sold during a week is modelled by a beta distribution with $\alpha = 2$ and $\beta = 2$, so

$$f(y) = \begin{cases} k\,y(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find k. ______________

(b) Find the probability that the station sells more than 93% of its stock in a given week, $P(Y > 0.93)$, to at least four significant figures.

____

(c) Find the expected quantity of petrol still in the tanks when they are refilled, in thousand litres. ______________

(a) $k = 1/B(2,2) = \Gamma(4)/(\Gamma(2)\Gamma(2)) = (3)!/(1)! = 3 \cdot 2 = 6$.

(b) As in Example 4.11, integrate the polynomial directly. For $0 \le y \le 1$,

$$F(y) = 6\int_0^y \left(t^1 - t^2\right)dt = 6\left(\frac{y^2}{2} - \frac{y^3}{3}\right) = 3\,y^2 - 2\,y^3.$$

Then $F(0.93) = 3(0.93)^2 - 2(0.93)^3 = 0.9860$ and

$$P(Y > 0.93) = 1 - 0.9860 = 0.0140.$$

(c) The unsold quantity is $35(1 - Y)$ thousand litres. By Theorem 4.11, $E(Y) = 2/4 = 0.5000$, so the expected unsold quantity is $35(1 - 0.5000) = 17.500$ thousand litres.

Problem 6

seed 20260925

Section 4.8 notes that a model is often chosen from **theoretical considerations**, the way the random quantity arises, before any data are seen. For each quantity below, choose the continuous model that those considerations point to.

(a) The quantity is the rounding error, in qapik, when a currency-exchange office rounds every payout to the nearest 10 qapik (the error lies between -5 and 5 and no value in that range is favoured).

- ○ uniform
- ○ normal
- ○ exponential
- ○ gamma with alpha > 1
- ○ beta

(b) The quantity is the proportion of a Baku hotel's rooms that are occupied on a given night.

- ○ uniform
- ○ normal
- ○ exponential

○ gamma with alpha > 1
○ beta

(c) Claims reach a motor insurer's call centre in Baku as a Poisson process at an average rate of 5 claims per hour. Let W be the time, in hours, between the arrival of claim number 4 and claim number 8 of the day. Give $E(W)$ and $V(W)$ to at least four significant figures.

$E(W) =$ ______________ hours, $V(W) =$ ______________ hours^2

(a) uniform: every value in a bounded interval is equally likely, which is the uniform density of Section 4.4.

(b) beta: it is a proportion confined to the interval [0,1], the use for which Section 4.7 introduces the beta density.

(c) Section 4.8: for a Poisson process, the time between the ath and bth events has a gamma distribution with $\alpha = b - a$. Here $\alpha = 8 - 4 = 4$; only the number of claims spanned matters, not which claim we start from. The gaps between successive claims are exponential with mean $\beta = 1/5 = 0.2000$ hours, so W is gamma with $\alpha = 4$ and $\beta = 1/5$. By Theorem 4.8,

$$E(W) = \alpha\beta = \frac{4}{5} = 0.8000, \qquad V(W) = \alpha\beta^2 = \frac{4}{5^2} = 0.1600.$$

Problem 7

seed 20260925

A fintech company runs QR-code payments for small merchants in Baku. Its share Y of card-and-QR payment volume at small merchants next year is modelled by a beta distribution with $\alpha = 3$ and $\beta = 7$:

$$f(y) = \frac{y^2(1-y)^6}{B(3,7)}, \qquad 0 \le y \le 1.$$

Section 4.7 shows that when α and β are positive integers,

$$F(y) = \sum_{i=\alpha}^{n} \binom{n}{i} y^i (1-y)^{n-i}$$

for some n, a binomial probability with success probability $p = y$.

(a) What is n here? ______________

(b) The probability that the fintech's share is at most 43%, $P(Y \le 0.43)$: ______________

(c) The probability that its share exceeds 43%: ______________

(d) $E(Y) =$ ______________

Give probabilities to at least four significant figures.

(a) $n = \alpha + \beta - 1 = 3 + 7 - 1 = 9$.

(b) Let W be binomial with $n = 9$ and $p = 0.43$. Then $F(0.43) = P(W \ge 3)$, the sum of the binomial probabilities for $i = 3, \ldots, 9$:

$$F(0.43) = 0.2291 + 0.2592 + 0.1955 + 0.0983 + 0.0318 + 0.0060 + 0.0005 = 0.8204.$$

(Equivalently $1 - P(W \le 2)$, summing the terms $i = 0, \ldots, 2$, which is quicker when α is small.)

(c) $P(Y > 0.43) = 1 - F(0.43) = 0.1796$, which is $P(W \le 2)$: the share exceeds y exactly when fewer than α of n Bernoulli(y) trials succeed.

(d) Theorem 4.11: $E(Y) = 3/10 = 0.3000$.

The relation is what makes integer-parameter beta probabilities practical by hand: integrating $y^2(1-y)^6$ means expanding a polynomial, while the binomial sum can be read from Table 1 or computed with pbinom in R.

Problem 8

seed 20260925

A cement plant near Qaradag can produce 90 thousand tonnes a month. Its capacity utilisation Y (output as a fraction of capacity) varies with construction demand and is modelled as beta with $\alpha = 3$ and $\beta = 4$. Each tonne sold contributes 42 AZN above variable cost, and the plant's fixed costs are 1663.2 thousand AZN a month, so monthly profit, in thousand AZN, is

$$P = 3780\,Y - 1663.2.$$

Give every answer to at least four significant figures.

(a) $E(P) =$ ____________ thousand AZN

(b) The standard deviation of P: ____________ thousand AZN

(c) The probability that the plant makes a loss in a given month: ____________

By Theorem 4.11,

$$E(Y) = \frac{3}{7} = 0.4286, \qquad V(Y) = \frac{3 \cdot 4}{7^2(7+1)} = 0.030612, \qquad \sigma_Y = 0.1750.$$

(a) By Theorem 4.5, $E(P) = 3780\,E(Y) - 1663.2 = 3780 \times 0.4286 - 1663.2 = -43.20$.

(b) The constant 1663.2 shifts P without changing its spread, so $\sigma_P = 3780\,\sigma_Y = 3780 \times 0.1750 = 661.36$.

(c) A loss means $P < 0$, that is $Y < 1663.2/3780 = 0.4400$, the break-even utilisation. With integer parameters, Section 4.7 gives $F(y) = P(W \geq \alpha)$ for W binomial with $n = \alpha + \beta - 1 = 6$ and $p = y$:

$$P(\text{loss}) = F(0.4400) = \sum_{i=3}^{6} \binom{6}{i} (0.4400)^i (1 - 0.4400)^{6-i} = 0.5382.$$

Equivalently $1 - P(W \leq 2)$.

Problem 9

seed 20260925

An investment committee in Baku compares three rebalancing rules for a balanced fund. Under each rule the proportion Y of the fund held in equities at a month-end is modelled by a symmetric beta distribution, $\alpha = \beta = k$, centred on the 50% target:

- Rule A: $k = 8$
- Rule B: $k = 6$
- Rule C: $k = 1$

(a) Give the standard deviation of Y under each rule, to at least four significant figures.

Rule A: ____________, Rule B: ____________, Rule C: ____________

(b) Which rule keeps the equity proportion most tightly around its target?

○ Rule A
○ Rule B
○ Rule C

(c) Under Rule C ($k = 1$), find the probability that the equity proportion is within 0.22 of the target, $P(0.28 < Y < 0.72)$:

____

(a) With $\alpha = \beta = k$, Theorem 4.11 gives $E(Y) = k/(2k) = 1/2$ for every rule and

$$V(Y) = \frac{k \cdot k}{(2k)^2(2k+1)} = \frac{1}{4(2k+1)}.$$

Rule A: $V = 1/(4 \cdot 17) = 0.014706$, $\sigma = 0.1213$. Rule B: $V = 1/(4 \cdot 13) = 0.019231$, $\sigma = 0.1387$. Rule C: $V = 1/(4 \cdot 3) = 0.083333$, $\sigma = 0.2887$.

(b) All three have the same mean, so the rule with the smallest standard deviation, the largest k, concentrates the equity share most tightly around 50%: Rule A. This is the pattern of Exercise 4.132: as $\alpha = \beta$ grows the symmetric beta density becomes taller and narrower around $1/2$.

(c) For $k = 1$, use the beta-binomial relation of Section 4.7 at each endpoint: $F(0.72) = 0.7200$ and $F(0.28) = 0.2800$, so

$$P(0.28 < Y < 0.72) = 0.7200 - 0.2800 = 0.4400.$$

With k = 1 this is the uniform distribution on (0, 1) (Exercise 4.127), so the answer is simply the length of the interval.

Problem 10

seed 20260925

A bank's card-risk team tracks **utilisation**: a customer's end-of-month balance divided by the card's credit limit. Across its credit-card book, utilisation Y is modelled as beta. Historical data give

$$E(Y) = 0.200000, \qquad V(Y) = 0.014545.$$

(a), **(b)** Find the beta parameters that reproduce this mean and variance. (They should come out as whole numbers, up to rounding in the figures above.)

$\alpha =$ ______________ , $\beta =$ ______________

(c) Customers with utilisation above 49% are flagged for a limit review. Using your (whole-number) α and β, find the probability that a randomly chosen customer is flagged, to at least four significant figures. ______________

(a), (b) Write $\mu = \alpha/(\alpha+\beta)$ and $c = \alpha+\beta$. Theorem 4.11 gives

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\mu(1-\mu)}{c+1}, \qquad \text{so} \qquad c = \frac{\mu(1-\mu)}{\sigma^2} - 1.$$

Here $\mu(1-\mu) = 0.160000$ and $c = 0.160000/0.014545 - 1 = 10.0003$, so

$$\alpha = \mu c = 2, \qquad \beta = (1-\mu)c = 8.$$

(c) With $\alpha = 2$, $\beta = 8$ and $n = \alpha+\beta-1 = 9$, Section 4.7 gives $F(y) = P(W \geq 2)$ for W binomial($9, y$). So

$$P(Y > 0.49) = 1 - F(0.49) = P(W \leq 1) = \sum_{i=0}^{1} \binom{9}{i} (0.49)^i (1-0.49)^{9-i} = 0.0225.$$

Problem 11

seed 20260925

For each quantity in (a) and (b), choose the continuous model that the way the quantity arises (Section 4.8) points to.

(a) The quantity is the number of minutes into a 60-minute settlement window at which a bond trade settles, when settlement is equally likely at any instant of the window.

- ◯ uniform
- ◯ normal
- ◯ exponential
- ◯ gamma with alpha > 1
- ◯ beta

(b) The quantity is the share of a household's savings that it keeps in foreign currency.

- ◯ uniform
- ◯ normal
- ◯ exponential
- ◯ gamma with alpha > 1
- ◯ beta

(c) A bank's treasury expects the spread Y between its lending rate and the interbank rate next quarter to lie between 1.25 and 5.25 percentage points. It models the position of the spread within that range,

$$Y^* = \frac{Y - 1.25}{4.00},$$

as beta with $\alpha = 3$ and $\beta = 5$ (Section 4.7 notes that a beta model applies to any interval $c \le y \le d$ through this change of variable).

Give each answer to at least four significant figures.

$E(Y) =$ ______________ percentage points

standard deviation of Y: ______________ percentage points

$P(Y > 3.250)$, the probability the spread is in the upper half of the range: ______________

(a) uniform: every value in a bounded interval is equally likely, so the uniform density of Section 4.4 is the model.

(b) beta: it is a proportion confined to [0,1], which is what the beta density of Section 4.7 is for.

(c) Write $Y = 1.25 + 4.00\,Y^*$. Theorem 4.11 gives $E(Y^*) = 3/8 = 0.3750$ and $\sigma_{Y^*} = \sqrt{3 \cdot 5/(8^2 \cdot (8+1))} = 0.1614$. By Theorem 4.5,

$$E(Y) = 1.25 + 4.00 \times 0.3750 = 2.7500, \qquad \sigma_Y = 4.00 \times 0.1614 = 0.6455.$$

Adding the constant 1.25 moves the distribution without spreading it.

The event $Y > 3.250$ is the event $Y^* > 0.5$. With integer parameters, $F(0.5) = P(W \ge 3)$ for W binomial with $n = 7$ and $p = 0.5$, so

$$P(Y > 3.250) = 1 - F(0.5) = P(W \le 2) = 0.2266.$$

Because alpha < beta the model leans towards the bottom of the range, so this is below 1/2.

Problem 12

seed 20260925

A telecom operator is launching a home-internet service in a region with 220 thousand households. Its eventual market share Y is modelled as beta with $\alpha = 2$ and $\beta = 8$. To win a larger share it must cut its tariff: at share Y the monthly tariff is $13 - 5.5Y$ AZN (so between 7.5 and 13 AZN). Monthly revenue, in thousand AZN, is

$$R = 220\,Y\,(13 - 5.5\,Y).$$

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________

(b) $E(Y^2) =$ ______________

(c) $E(R) =$ ______________ thousand AZN

(d) A planner plugs the expected share straight into the revenue formula, computing $220\,\mu\,(13 - 5.5\,\mu)$ with $\mu = E(Y)$. What figure does the planner get, in thousand AZN? ______________

(a) Theorem 4.11: $E(Y) = 2/10 = 0.2000$.

(b) From Theorem 4.11, $V(Y) = \alpha\beta/((\alpha+\beta)^2(\alpha+\beta+1)) = 0.014545$, and $E(Y^2) = V(Y) + [E(Y)]^2$. The algebra simplifies to

$$E(Y^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} = \frac{2 \cdot 3}{10 \cdot 11} = 0.0545.$$

(Directly: $E(Y^2) = B(\alpha+2, \beta)/B(\alpha, \beta)$, and the gamma functions cancel to the same fraction.)

(c) $R = 220(13\,Y - 5.5\,Y^2)$, so by Theorem 4.5

$$E(R) = 220\big(13\,E(Y) - 5.5\,E(Y^2)\big) = 220(13 \times 0.2000 - 5.5 \times 0.0545) = 506.000.$$

(d) $220 \times 0.2000 \times (13 - 5.5 \times 0.2000) = 523.600$.

The planner overstates expected revenue by $523.600 - 506.000 = 17.600$, which is exactly $220 \times 5.5 \times V(Y)$: because R is not linear in Y, $E(R) \ne R(E(Y))$, and the uncertainty in the share costs revenue.

Problem 13

seed 20260925

The container terminal at the Port of Baku in Alat reports its daily berth utilisation Y, the fraction of berth-hours in use. The port authority models it with

$$f(y) = \begin{cases} k\,y(1-y)^2, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Operations are “normal” when utilisation is between 47% and 88%, and the terminal is “congested” above 88%.

Give probabilities to at least four significant figures.

(a) $k =$ ______________

(b) The probability that a given day has normal operations, $P(0.47 \le Y \le 0.88)$: ______________

(c) Given that utilisation on a day is above 47%, the probability that the terminal is congested: ______________

(a) This is a beta density with $\alpha = 2$, $\beta = 3$, so $k = \Gamma(5)/(\Gamma(2)\Gamma(3)) = 12$.

(b) Integrating the polynomial term by term (or using the binomial form of Section 4.7 with $n = 4$) gives, for $0 \le y \le 1$,

$$F(y) = 6y^2 - 8y^3 + 3y^4.$$

Then $F(0.88) = 0.9937$, $F(0.47) = 0.6412$ and, by Theorem 4.3,

$$P(0.47 \le Y \le 0.88) = F(0.88) - F(0.47) = 0.3525.$$

(c) By Definition 2.9, with $\{Y > 0.88\} \subset \{Y > 0.47\}$,

$$P(Y > 0.88 \mid Y > 0.47) = \frac{P(Y > 0.88)}{P(Y > 0.47)} = \frac{0.0063}{0.3588} = 0.0175.$$

Problem 14

seed 20260925

A market maker on the Baku Stock Exchange quotes a thinly traded corporate bond. Two analysts model the waiting time T, in minutes, from a quote until the next trade. Both agree that $E(T) = 7$ minutes.

- Analyst U uses a uniform distribution on $(0, 14)$.
- Analyst E uses an exponential distribution with mean $\beta = 7$.

Give every answer to at least four significant figures.

(a) $P(T > 11.9)$ under Analyst U’s model: ______________

(b) $P(T > 11.9)$ under Analyst E’s model: ______________

(c) $V(T)$ under each model. U: ______________, E: ______________ minutes^2

(d) The market maker’s quote goes stale, and must be pulled, if no trade occurs within 16.8 minutes. Under Analyst U’s model that never happens. What is $P(T > 16.8)$ under Analyst E’s model? ______________

Check first that both models have the stated mean: by Theorem 4.6 the uniform on $(0, 14)$ has mean $14/2 = 7$, and by Theorem 4.10 the exponential with $\beta = 7$ has mean 7.

(a) For the uniform, $P(T > t) = (14 - t)/14$ on $(0, 14)$, so $P(T > 11.9) = 1 - 11.9/14 = 0.1500$.

(b) For the exponential, $P(T > t) = \int_t^\infty \frac{1}{7} e^{-y/7}\,dy = e^{-t/7}$, so $P(T > 11.9) = e^{-1.7} = 0.1827$.

(c) Theorem 4.6: $V(T) = (14 - 0)^2/12 = 16.3333$. Theorem 4.10: $V(T) = \beta^2 = 49.0000$, three times as large.

(d) $P(T > 16.8) = e^{-2.4} = 0.0907$, whereas the uniform model puts probability zero beyond 14 minutes.

This is the question Section 4.8 raises: how much does it matter if we use the wrong density? Two models that agree on the mean can disagree sharply on the tail, and a risk decision such as how often a quote goes stale lives in the tail. If trades arrive as a Poisson process, Section 4.8's theoretical argument favours the exponential.

Problem 15

seed 20260925

A bank's analyst first models the recovery rate Y on defaulted SME loans (the fraction of the balance eventually recovered) as normal with $\mu = 0.7540$ and $\sigma = 0.10$. A recovery rate cannot exceed 1 or fall below 0.

(a) Using Table 4, find the probability the normal model assigns to recovering **more than** the whole balance, $P(Y > 1)$, to four decimal places. ______________

(b) Using Table 4, find $P(Y < 0)$ under the normal model, to four decimal places. ______________

(c) The analyst switches to a beta model with the same mean and variance. Find its parameters, to at least four significant figures.

$\alpha =$ ______________, $\beta =$ ______________

(a) Standardise: $z = (1 - 0.7540)/0.10 = 2.46$, so $P(Y > 1) = P(Z > 2.46) = 0.0069$ from Table 4.

(b) $z = (0 - 0.7540)/0.10 = -7.54$, so by symmetry $P(Y < 0) = P(Z > 7.54) = 0.0000$. This z-score is beyond the main body of Table 4; the tail area is below P(Z > 3.0) = 0.00135, so to four decimal places it is essentially zero.

The normal model therefore puts about 0.0069 of its probability on recovery rates above 100%, which is impossible. Section 4.8 says a model need not fit perfectly, but a model whose support includes impossible values invites wrong inferences exactly in the tails a risk manager cares about.

(c) Invert Theorem 4.11. With $c = \alpha + \beta$, $\sigma^2 = \mu(1-\mu)/(c+1)$, so

$$c = \frac{\mu(1-\mu)}{\sigma^2} - 1 = \frac{0.185484}{0.010000} - 1 = 17.5484,$$
$$\alpha = \mu c = 13.2315, \qquad \beta = (1-\mu)c = 4.3169.$$

This beta model has exactly the analyst's mean and variance but lives on $[0, 1]$.

Problem 16

seed 20260925

An Azerbaijani property insurer buys reinsurance that pays out only from the 2th large hail or flood claim onwards. Large claims arrive as a Poisson process at an average rate of 1.9 per month. Let T be the time, in months from the start of the contract, until the 2th large claim.

Give every answer to at least four significant figures.

(a) $E(T) =$ ______________ months

(b) $V(T) =$ ______________ months^2

(c) The contract runs for 7 months. Find the probability that the reinsurance is triggered during the contract, $P(T \le 7)$. ______________

Section 4.8: for a Poisson process, the time from the start until the kth event (the time between the 0th and kth events) has a gamma distribution with $\alpha = k = 2$; the scale is the mean gap between events, $\beta = 1/1.9 = 0.5263$ months.

(a), (b) Theorem 4.8: $E(T) = \alpha\beta = 2/1.9 = 1.0526$ and $V(T) = \alpha\beta^2 = 2/1.9^2 = 0.5540$.

(c) The reinsurance is triggered by month 7 exactly when at least 2 claims occur in 7 months. The number of claims N in 7 months is Poisson with mean $\lambda = 1.9 \times 7 = 13.3$, so

$$P(T \le 7) = P(N \ge 2) = 1 - \sum_{j=0}^{1} \frac{e^{-13.3}(13.3)^j}{j!} = 1 - 0.0000 = 1.0000.$$

This is the gamma analogue of the beta-binomial relation of Section 4.7: with an integer shape parameter, the continuous cdf equals a discrete tail sum. Integrating the gamma density by parts 1 times gives the same answer.

Problem 17

seed 20260925

During a week, the proportion of time Y that a gas-fired power unit near Mingachevir is offline for maintenance or repair has a beta distribution with $\alpha = 1$ and $\beta = 6$, so $f(y)$ is proportional to $(1-y)^5$ on $[0, 1]$. The weekly cost of downtime, in thousand AZN, is

$$C = 11 + 37\,Y + 7\,Y^2,$$

where the quadratic term reflects emergency power purchases that become more expensive the longer the unit is out.

Give every answer to at least four significant figures, and keep at least six significant figures in intermediate results (differences of close numbers appear below).

(a) $E(Y^2) =$ ______________

(b) $E(C) =$ ______________

(c) $V(C) =$ ______________

Beta moments. For any $k > 0$,

$$E(Y^k) = \int_0^1 y^k\,\frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}\,dy = \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+k)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta+k)},$$

the same step as in the proof of Theorem 4.11. Using $\Gamma(x+1) = x\Gamma(x)$ this is a product: $E(Y^k) = \prod_{j=0}^{k-1} \frac{\alpha+j}{\alpha+\beta+j}$. With $\alpha + \beta = 7$:

$E(Y) = \frac{1}{7} = 0.142857, \quad E(Y^2) = E(Y)\cdot\frac{2}{8} = 0.035714, \quad E(Y^3) = E(Y^2)\cdot\frac{3}{9} = 0.011905, \quad E(Y^4) = E(Y^3)\cdot\frac{4}{10} = 0.004762.$

(a) $E(Y^2) = 0.035714$.

(b) By Theorem 4.5, $E(C) = 11 + 37\,E(Y) + 7\,E(Y^2) = 11 + 37(0.142857) + 7(0.035714) = 16.5357$.

(c) Expand the square and apply Theorem 4.5 term by term:

$$E(C^2) = 11^2 + 37^2E(Y^2) + 7^2E(Y^4) + 2(11)(37)E(Y) + 2(11)(7)E(Y^2) + 2(37)(7)E(Y^3) = 298.0786.$$

Then $V(C) = E(C^2) - [E(C)]^2 = 298.0786 - 16.5357^2 = 24.6487$ (standard deviation 4.9647 thousand AZN).

A tidier route: the constant 11 does not affect the variance, and

$$V(C) = 37^2\,V(Y) + 7^2\,V(Y^2) + 2(37)(7)\left[E(Y^3) - E(Y)E(Y^2)\right]$$

with $V(Y) = 0.015306$, $V(Y^2) = E(Y^4) - [E(Y^2)]^2 = 0.003486$ and $E(Y^3) - E(Y)E(Y^2) = 0.006803$, which gives the same 24.6487. Either way the fourth moment is needed, because C contains Y^2.

Problem 18

seed 20260925

A bank in Baku holds a 1800 thousand AZN loan to a logistics firm. Its credit model has two pieces:

- the time τ until the firm defaults is exponential with mean 14 years;
- if the firm defaults, the recovery rate R is beta with $\alpha = 3$ and $\beta = 5$, independent of when default happens.

The **loss given default** is $\mathrm{LGD} = 1 - R$, and the loss on the loan over the next 5 years is $L = 1800 \cdot D \cdot \mathrm{LGD}$, where $D = 1$ if $\tau \le 5$ and $D = 0$ otherwise.

Give every answer to at least four significant figures, and keep at least six significant figures in intermediate results (differences of close numbers appear below).

(a) The probability of default within 5 years, $P(\tau \le 5)$: ______________

(b) $E(\mathrm{LGD}) =$ ______________

(c) The probability that, given default, the bank loses more than 59% of the balance, $P(\mathrm{LGD} > 0.59)$: ______________

(d) The expected loss $E(L)$, in thousand AZN: ______________

(e) The standard deviation of L, in thousand AZN: ______________

(a) The exponential with mean $\beta = 14$ has $F(t) = 1 - e^{-t/14}$, so $P(\tau \le 5) = 1 - e^{-0.3571} = 0.300327$.

(b) If R is beta(α, β) then $\text{LGD} = 1 - R$ is beta with the parameters swapped, $\alpha^* = 5$ and $\beta^* = 3$ (Exercise 4.114e): substituting $r = 1 - \ell$ turns $r^{\alpha-1}(1-r)^{\beta-1}$ into $\ell^{\beta-1}(1-\ell)^{\alpha-1}$, and $B(\alpha, \beta) = B(\beta, \alpha)$. By Theorem 4.11, $E(\text{LGD}) = 5/8 = 0.6250$ (equivalently $1 - E(R)$).

(c) With LGD beta(5, 3) and $n = 7$, Section 4.7 gives $F(\ell) = P(W \ge 5)$ for W binomial(7, ℓ), so

$$P(\text{LGD} > 0.59) = 1 - F(0.59) = P(W \le 4) = \sum_{i=0}^{4} \binom{7}{i} (0.59)^i (1 - 0.59)^{7-i} = 0.6017.$$

(d) D takes the value 1 with probability $p = 0.300327$, and D and LGD are independent, so the expectation of the product is the product of expectations:

$$E(L) = 1800 \cdot p \cdot E(\text{LGD}) = 1800 \times 0.300327 \times 0.6250 = 337.8684.$$

This is the familiar “EL = EAD × PD × LGD” of bank capital rules.

(e) Since $D^2 = D$, $L^2 = 1800^2\, D\, \text{LGD}^2$, and $E(\text{LGD}^2) = \alpha^*(\alpha^*+1)/((\alpha^*+\beta^*)(\alpha^*+\beta^*+1)) = 5 \cdot 6/(8 \cdot 9) = 0.416667$. So $E(L^2) = 1800^2 \times 0.300327 \times 0.416667 = 405442.0745$ and

$$\sigma_L = \sqrt{E(L^2) - [E(L)]^2} = \sqrt{405442.0745 - 337.8684^2} = 539.7101.$$

The standard deviation is large relative to the expected loss: a single loan either loses nothing or loses a sizeable share of its balance.

Problem 19

seed 20260925

A central-bank study of household credit models the **debt-service ratio** Y, the share of a household’s monthly income spent on loan repayments, as beta with $\alpha = 2$ and $\beta = 7$:

$$f(y) = \frac{y^1(1-y)^6}{B(2,7)}, \qquad 0 \le y \le 1.$$

The ratio $Y/(1-Y)$ compares repayments with the income that is left after them.

Use $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(x+1) = x\,\Gamma(x)$. Give every answer to at least four significant figures.

(a) $B(2,7) =$ ______________

(b) $E(Y) =$ ______________. For a household with monthly income 1650 AZN, the expected monthly repayment is ______________ AZN.

(c) $E\left(\frac{Y}{1-Y}\right) =$ ______________

(d) $E\left[\left(\frac{Y}{1-Y}\right)^2\right] =$ ______________

(a) By Definition 4.12, $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. Build each gamma value from $\Gamma(1) = 1$ or $\Gamma(1/2) = \sqrt{\pi}$ with $\Gamma(x+1) = x\Gamma(x)$: $\Gamma(2) = 1.000000$, $\Gamma(7) = 720.000000$, $\Gamma(9) = 40320.000000$, so

$$B(2,7) = \frac{1.000000 \times 720.000000}{40320.000000} = 0.0178571,$$

and the density’s constant is $1/B = 56.0000$.

(b) Theorem 4.11: $E(Y) = 2/9 = 0.2222$; expected repayment $1650 \times 0.2222 = 366.67$ AZN.

(c) By Theorem 4.4,

$$E\left(\frac{Y}{1-Y}\right) = \frac{1}{B(\alpha,\beta)} \int_0^1 y^{\alpha}(1-y)^{\beta-2}\, dy = \frac{B(\alpha+1, \beta-1)}{B(\alpha,\beta)}.$$

The integral is a beta integral because $\beta - 1 > 0$. Writing both beta functions with gammas, $\Gamma(\alpha+\beta)$ cancels, and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\Gamma(\beta) = (\beta-1)\Gamma(\beta-1)$ give

$$E\left(\frac{Y}{1-Y}\right) = \frac{\alpha}{\beta-1} = \frac{2}{6} = 0.3333.$$

(d) In the same way, with $\beta - 2 > 0$,

$$E\left[\left(\frac{Y}{1-Y}\right)^2\right] = \frac{B(\alpha+2,\beta-2)}{B(\alpha,\beta)} = \frac{\alpha(\alpha+1)}{(\beta-1)(\beta-2)} = \frac{2\times 3}{6\times 5} = 0.2000.$$

So the odds have variance $0.2000 - 0.3333^2 = 0.0889$. Note that $E(Y/(1-Y)) \neq E(Y)/(1-E(Y))$: the expectation of a ratio is not the ratio of expectations. If $\beta \leq 2$ the integral in (d) diverges, because the odds explode as $y \to 1$.

Problem 20

seed 20260925

Azerbaijan's grid operator plans for the evening peak hour. Let Y be peak demand as a fraction of the grid's peak capacity of 1900 MW, modelled as beta with $\alpha = 3$ and $\beta = 4$. Domestic plants can cover demand up to a fraction $c = 0.72$ of capacity; any excess, $(Y-c)^+ = \max(Y-c, 0)$ times 1900 MW, must be imported for the hour at 105 AZN per MWh. So the import cost for the peak hour, in AZN, is

$$C = 105 \times 1900 \times (Y - 0.72)^+.$$

Give every answer to at least four significant figures, and keep at least six significant figures in intermediate results (differences of close numbers appear below).

(a) The probability that any import is needed, $P(Y > 0.72)$: ____________

(b) $E\big[(Y-0.72)^+\big] =$ ____________

(c) $E(C) =$ ____________ AZN

(d) A second analyst uses the uniform model on $(0,1)$ (the beta with $\alpha = \beta = 1$). What value of $E\big[(Y-0.72)^+\big]$ does that model give?

____

(a) With $n = \alpha + \beta - 1 = 6$, Section 4.7 gives $F(y) = P(W \geq 3)$ for W binomial(6, y). Here $F(0.72) = 0.9443$, so $P(Y > 0.72) = 0.0557$.

(b) By Theorem 4.4,

$$E\big[(Y-c)^+\big] = \int_c^1 (y-c)f(y)\,dy = \int_c^1 yf(y)\,dy - c\,P(Y > c).$$

For the first integral use the step in the proof of Theorem 4.11:

$$y\,\frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \cdot \frac{y^{\alpha}(1-y)^{\beta-1}}{B(\alpha+1,\beta)} = \mu\, f_{\alpha+1,\beta}(y),$$

where $\mu = \alpha/(\alpha+\beta) = 0.4286$ and $f_{\alpha+1,\beta}$ is the beta(4, 4) density. Hence $\int_c^1 yf(y)\,dy = \mu\,[1 - F_{4,4}(c)]$, and the beta-binomial relation with $n = 7$ gives $F_{4,4}(0.72) = 0.8984$:

$$\int_{0.72}^1 yf(y)\,dy = 0.428571(1 - 0.898404) = 0.043541, \qquad E\big[(Y-0.72)^+\big] = 0.043541 - 0.72 \times 0.055712 = 0.003428.$$

(Expanding $(y-c)\,y^2(1-y)^3$ and integrating term by term gives the same number, with more algebra.)

(c) By Theorem 4.5, $E(C) = 105 \times 1900 \times 0.003428 = 199500 \times 0.003428 = 683.95$ AZN.

(d) With $f(y) = 1$ on $(0,1)$, $E\big[(Y-c)^+\big] = \int_c^1 (y-c)\,dy = (1-c)^2/2 = 0.28^2/2 = 0.03920$, which would put the expected import bill at 7820.40 AZN, higher than the beta model's 683.95 AZN. Section 4.8's question, how much the wrong model matters, has a sharp answer here: a cost that depends only on the upper tail is sensitive to the tail the model assumes.

Answer key

Problem	Answers
1	(a) 0.25, (b) 0.0208333, (c) 105, (d) 60.6218

2	(a) 56, (b) 2, (c) 7, (d) 0.222222, (e) 71.1111
3	(a) 0.761672, (b) 0.520858, (c) 0.25
4	(a) y^4, (b) 0.251948, (c) 0.840896, (d) 176
5	(a) 6, (b) 0.014014, (c) 17.5
6	(a) Choice 1, (b) Choice 5, (c) 0.8, (d) 0.16
7	(a) 9, (b) 0.8204, (c) 0.1796, (d) 0.3
8	(a) -43.2, (b) 661.362, (c) 0.538172
9	(a) 0.121268, (b) 0.138675, (c) 0.288675, (d) Rule A, (e) 0.44
10	(a) 2.00007, (b) 8.00028, (c) 0.0225178
11	(a) Choice 1, (b) Choice 5, (c) 2.75, (d) 0.645497, (e) 0.226562
12	(a) 0.2, (b) 0.0545455, (c) 506, (d) 523.6
13	(a) 12, (b) 0.352504, (c) 0.0175307
14	(a) 0.15, (b) 0.182684, (c) 16.3333, (d) 49, (e) 0.090718
15	(a) 0.00694685, (b) 0, (c) 13.2315, (d) 4.31691
16	(a) 1.05263, (b) 0.554017, (c) 0.999976
17	(a) 0.0357143, (b) 16.5357, (c) 24.6487
18	(a) 0.300327, (b) 0.625, (c) 0.60168, (d) 337.868, (e) 539.71
19	(a) 0.0178571, (b) 0.222222, (c) 366.667, (d) 0.333333, (e) 0.2
20	(a) 0.0557124, (b) 0.00342831, (c) 683.947, (d) 0.0392