Week 11, Problem Set 2

Wackerly 4.9-4.11 - Other Expected Values, Tchebysheff's Theorem, and Chapter 4 in Review

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

After a motor-insurance claim is approved, the delay Y (in days) until the insurer actually pays is modelled by the exponential density

$$f(y) = \begin{cases} \dfrac{1}{16}\, e^{-y/16}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Use Definition 4.14 to find the moment-generating function of Y: $m(t) =$ ______________ (a formula in t)

(b) The integral in (a) is finite only for $t < b$. Find b: ______________

(c) Differentiate $m(t)$ to find $E(Y) =$ ______________ days

(d) and $V(Y) =$ ______________ (days squared).

(e) A court order makes the insurer pay penalty interest compounded continuously at 0.95% per day of delay, so a claim of 1 AZN is settled as $e^{0.0095\,Y}$ AZN. Find the expected settlement multiplier $E\left(e^{0.0095\,Y}\right) =$

____

Give numerical answers to at least four significant figures.

Problem 2

seed 20260925

A broker splits client orders into blocks. The size V of a block (in thousand shares of a Baku-listed bank) has moment-generating function

$$m(t) = (1 - 2.8\,t)^{-3}, \qquad t < 1/2.8.$$

(a) $E(V) =$ ______________

(b) $E(V^2) =$ ______________

(c) $V(V) =$ ______________

(d) The third moment about the origin, $\mu_3' = E(V^3) =$ ______________

(e) Executing a block of size V costs $C = 45\,V + 9\,V^2$ AZN: commission plus a market-impact term that grows with the square of the block. Find $E(C) =$ ______________ AZN

Give answers to at least four significant figures.

Problem 3

seed 20260925

A bank's stress test models three quantities for the coming quarter, all in million AZN. Their moment-generating functions are

$$m_1(t) = (1 - 2.9\,t)^{-2}, \qquad m_2(t) = \frac{1}{1 - 3.00\,t}, \qquad m_3(t) = e^{7t + 8.50\,t^2}.$$

X_1 is the operational loss, X_2 the loss on a single large exposure and X_3 the trading profit-and-loss.

(a) X_1 has a gamma distribution with $\alpha =$ ____________ and $\beta =$ ____________

(b) X_2 is exponential. Its mean is ____________ and its standard deviation is ____________

(c) X_3 is normal with mean $\mu =$ ____________ and variance $\sigma^2 =$ ____________

(d) Which quantity has the largest variance?

- ○ The loss X_1
- ○ The loss X_2
- ○ The trading profit-and-loss X_3

Problem 4

seed 20260925

(a) A mobile operator in Baku wants to advertise a range of monthly data use that excludes no more than 12% of its subscribers. All it knows is that monthly use has mean 12 GB and standard deviation 1.92 GB; the shape of the distribution is unknown. Using Tchebysheff's theorem, the narrowest interval of the form $(\mu - k\sigma,\ \mu + k\sigma)$ that is guaranteed to do this is

from ____________ GB to ____________ GB.

(b) A bank's cash-logistics team wants the daily withdrawals Y at a branch ATM to fall within 36 thousand AZN of their mean μ on at least 82% of days. Again only μ and σ are known. What is the largest standard deviation σ for which Tchebysheff's theorem still guarantees this?

$\sigma_{\max} =$ ____________ thousand AZN

Give answers to at least four significant figures.

Problem 5

seed 20260925

The daily return Y (in percent) of a stock index is modelled as normal with mean $\mu = 0.03$ and standard deviation $\sigma = 1.9$. The risk desk flags a day as a "large move" when the return is at or below -3.58 or at or above 3.64.

(a) A large move is a deviation of at least $k\sigma$ from the mean. Find $k =$ ____________

(b) Suppose the desk did not trust the normal model and knew only μ and σ. Give Tchebysheff's upper bound for the probability of a large move: ____________ (at least four significant figures)

(c) Under the normal model, find the exact probability of a large move, using Table 4, to four decimal places:

____

Problem 6

seed 20260925

A company holds a revolving credit line with a limit of $\theta = 19$ million AZN. The amount Y drawn at the end of a quarter (million AZN) has density

$$f(y) = \begin{cases} \dfrac{2\,y}{19^2}, & 0 \le y \le 19, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) $\mu_1' = E(Y) =$ ______________

(b) $\mu_2' = E(Y^2) =$ ______________

(c) the second central moment $\mu_2 = V(Y) =$ ______________

(d) For the quarter the bank earns interest of 11% on the drawn amount and a commitment fee of 1% on the undrawn amount $19 - Y$. Find the bank's expected income from this line, in million AZN: ______________

Give answers to at least four significant figures.

Problem 7

seed 20260925

The overnight interbank rate R (percent) for a given day is modelled as uniformly distributed on the band $(4.75,\ 7.75)$. A bank that must borrow overnight has a funding cost of

$$C = 11 + 45\,R \quad \text{thousand AZN.}$$

(a) From Definition 4.14, find the moment-generating function of R for $t \neq 0$: $m_R(t) =$ ______________

(b) Use $m_{aY+b}(t) = e^{tb}\, m_Y(at)$ (Exercise 4.137) to find the MGF of the cost, for $t \neq 0$: $m_C(t) =$

____

(c) The MGF in (b) identifies the distribution of C: it is uniform on an interval (c_1, c_2) with $c_1 =$ ______________ and $c_2 =$ ______________

(d) $V(C) =$ ______________

Enter formulas in t, for example (e^(5*t)-e^(4*t))/t; give numbers to at least four significant figures.

Problem 8

seed 20260925

The daily return Y (percent) of a stock index is normal with mean $\mu = 0.1$ and standard deviation $\sigma = 2.4$, so (Exercise 4.138)

$$m_Y(t) = e^{\mu t + \sigma^2 t^2/2}.$$

A dealer is short the index and also earns a fixed carry, so its daily profit, in thousand AZN, is

$$X = 5.5 - 5\,Y.$$

(a) Find the moment-generating function of X: $m_X(t) =$ ______________ (a formula in t)

(b) $E(X) =$ ______________

(c) $V(X) =$ ______________

(d) The MGF shows that X is normal. Using Table 4, find to four decimal places the probability of a loss of 10.48 thousand AZN or worse, i.e. $P(X < -10.48)$: ______________

Problem 9

seed 20260925

A property insurer in Baku reinsures a portfolio abroad. The size Y of a claim, in thousand USD, is gamma distributed with $\alpha = 2$ and $\beta = 4.7$. Each claim is converted to manat at the fixed rate of 1.70 AZN per USD, and a handling fee of 0.7 thousand AZN is added, so the cost of a claim in thousand AZN is

$$C = 0.7 + 1.7Y.$$

(a) Find the moment-generating function of C: $m_C(t) =$ ______________ (a formula in t)

(b) $m_C(t)$ exists for $t < b$. Find b: ______________

(c) The converted claim $1.7Y$ (without the fee) is gamma distributed with $\alpha =$ ______________ and $\beta =$ ______________

(d) Use $m_C(t)$ to find $E(C) =$ ______________

(e) and $V(C) =$ ______________

Give numerical answers to at least four significant figures.

Problem 10

seed 20260925

An insurer's risk team simulates the time W (in days) between successive motor claims by drawing U uniformly on $(0, 1)$ and setting

$$W = g(U) = -2.0 \ln U.$$

(a) Use Theorem 4.12, $E\left[e^{tg(U)}\right] = \int_0^1 e^{tg(u)} f_U(u)\, du$, to find the moment-generating function of W: $m_W(t) =$ ______________ (a formula in t)

(b) The integral converges only for $t < b$. Find b: ______________

(c) What is the distribution of W?

- ◯ Exponential with mean 2.0
- ◯ Gamma with alpha = 2 and beta = 2.0
- ◯ Normal with mean 2.0 and variance 1
- ◯ Uniform on (0, 2.0)

(d) Find $P(W > 5.60)$: ______________

(e) Find the median of W, the value $\phi_{0.5}$ with $P(W \leq \phi_{0.5}) = 0.5$: ______________ days

Give numerical answers to at least four significant figures.

Problem 11

seed 20260925

The time Y (minutes) that an operator at a bank's call centre spends on a customer call is exponential with mean $\beta = 4.0$. Staffing is planned around the interval $\mu \pm k\sigma$ with $k = 2.85$.

(a) Find the upper end $\mu + k\sigma$ of the interval: ______________ minutes

(b) Find the exact probability $P(|Y - \mu| < k\sigma)$: ______________

(c) What lower bound does Tchebysheff's theorem give for the probability in (b)? ______________

(d) Find the exact probability that a call runs past the upper end of the interval, $P(Y \geq \mu + k\sigma)$:

____

Give answers to at least four significant figures.

Problem 12

seed 20260925

At a Baku bank, the time Y (hours of officer work) needed to process a small-business loan application has, from past experience, a gamma distribution with $\alpha = 2.8$ and $\beta = 2.8$. A newly hired officer takes 22.1 hours on an application.

(a) $\mu =$ ______________ hours and $\sigma =$ ______________ hours

(b) By how many standard deviations does 22.1 exceed the mean? $k =$ ______________

(c) Tchebysheff's upper bound for $P(|Y - \mu| \geq k\sigma)$ with this k: ______________

(d) The bank's quality-control rule flags a new officer for review when, under the past distribution, the probability of a deviation at least as large as the one observed is guaranteed to be at most 0.05. What does the rule say here?

- ◯ Flag: a deviation this large has probability at most 0.05 under past experience
- ◯ Do not flag: Tchebysheff's bound exceeds 0.05, so chance cannot be ruled out

Give numerical answers to at least four significant figures.

Problem 13

seed 20260925

A filling station on the Baku-Guba highway sells a random amount Y of diesel each day, in thousand litres, with density

$$f(y) = \begin{cases} \dfrac{3y^2}{3^3}, & 0 \leq y \leq 3, \\ 0, & \text{elsewhere.} \end{cases}$$

The station's margin is 85 AZN per thousand litres on days when $Y \leq 2.4$. On days when $Y > 2.4$ the supplier's volume rebate raises the margin to 140 AZN per thousand litres on the whole day's volume.

(a) Find $P(Y > 2.4)$: ______________

(b) Find the station's expected daily profit: ______________ AZN

(c) How much of the expected daily profit in (b) is due to the rebate, that is, the expected profit minus what it would be if the margin were always 85? ______________ AZN

Give answers to at least four significant figures.

Problem 14

seed 20260925

An importer's containers arriving at the Port of Baku take a customs-clearance time Y (days) that is uniformly distributed on $(1,\ 8.5)$. The terminal charges a flat 375 AZN per container for any clearance time up to 3.25 days. Beyond 3.25 days it charges 375 AZN plus 70 AZN per additional day, prorated: a clearance time of $3.25 + 1.5$ days costs $375 + 70(1.5)$ AZN.

(a) The probability that the importer pays more than the flat charge: ______________

(b) The expected cost per container: ______________ AZN

(c) The expected cost per container among containers that are not cleared within 3.25 days: ______________ AZN

Give answers to at least four significant figures.

Problem 15

seed 20260925

The damage Y (thousand AZN) in a motor accident reported to an insurer is exponential with mean $\beta = 1.8$. A policy has a deductible of $d = 1.4$: the insurer pays nothing when $Y \le d$ and $Y - d$ otherwise.

(a) The probability that a reported accident leads to a payment: ______________

(b) The expected payment per reported accident: ______________ thousand AZN

(c) The policy also has a limit: losses above $u = 11$ are covered only up to u, so the payment is $\min\{(Y-d)^+,\ u-d\}$, at most 9.6. Find the expected payment per reported accident now: ______________ thousand AZN

(d) With the deductible and the limit, the expected payment per accident that does lead to a payment: ______________ thousand AZN

Give answers to at least four significant figures.

Problem 16

seed 20260925

For a home-insurance policy in Baku, the total amount Y (thousand AZN) the insurer pays in a year is 0 with probability 0.75, since most households make no claim. Given that there is a claim, the amount paid has the exponential density

$$f(y) = \frac{1}{7.5} e^{-y/7.5}, \qquad y > 0.$$

(a) Find the distribution function of Y at 4.5: $F(4.5) =$ ______________

(b) Find $P(Y > 4.5)$: ______________

(c) Find $E(Y)$, the pure premium for the policy: ______________ thousand AZN

(d) Find $V(Y)$: ______________

Give answers to at least four significant figures.

Problem 17

seed 20260925

A trader holds a European call option on a share listed on the Baku Stock Exchange with strike price $K = 30.275$ AZN. Her model for the share price S at expiry is normal with mean $\mu = 32$ AZN and standard deviation $\sigma = 11.5$ AZN. The call pays $(S - K)^+ = \max(S - K, 0)$ at expiry.

(a) Using Table 4, the probability that the call finishes in the money, $P(S > K)$, to four decimal places:

____

(b) The expected payoff of the call, $E[(S - K)^+]$: ______________ AZN

(c) The expected payoff of the put with the same strike, which pays $(K - S)^+$: ______________ AZN

Hint: write $S = \mu + \sigma Z$ with Z standard normal, and use $\int_z^\infty u\,\phi(u)\,du = \phi(z)$, where $\phi(u) = e^{-u^2/2}/\sqrt{2\pi}$. Give (b) and (c) to at least four significant figures.

Problem 18

seed 20260925

A business-interruption policy sold to restaurants in Baku pays for income lost while a restaurant is closed. The annual loss Y (in ten thousand AZN) is 0 with probability 0.66. Given a loss, it has the gamma density with $\alpha = 2$ and $\beta = 4.7$:

$$f(y) = \frac{y\,e^{-y/4.7}}{4.7^2}, \qquad y > 0.$$

(a) $E(Y) =$ ______________

(b) $V(Y) =$ ______________

(c) The policy has a deductible of $d = 2.0$, so it pays $(Y - d)^+$. Find the probability that the policy pays anything in a year: ______________

(d) Find the expected annual payment $E[(Y - d)^+]$: ______________

(e) Find the expected payment in a year in which the policy does pay: ______________

Give answers to at least four significant figures.

Problem 19

seed 20260925

A bank models its annual operational-risk loss X (million AZN) as gamma with $\alpha = 2$ and $\beta = 3.8$, so $m_X(t) = (1 - 3.8t)^{-2}$. The regulator asks how likely a loss of $x_0 = 22.65$ million AZN or more is. A risk analyst rescales the loss to

$$U = \frac{2X}{\beta} = \frac{2X}{3.8}.$$

(a) Find the moment-generating function of U: $m_U(t) =$ ____________ (a formula in t)

(b) The MGF shows that U has a χ^2 distribution. With how many degrees of freedom? $\nu =$ ____________

(c) Using only the mean and variance of X, give Tchebysheff's upper bound for $P(X \geq 22.65)$, through $P(|X - \mu| \geq 22.65 - \mu)$: ____________

(d) Find the exact probability $P(X \geq 22.65)$: ____________

Give numerical answers to at least four significant figures.

Problem 20

seed 20260925

The pitch-control unit of a turbine at a wind farm on the Absheron peninsula has a life length Y (months) that is exponential with mean 21. The operator replaces the unit when it fails or when it reaches age $T = 34$ months, whichever comes first, so the unit is in use for $X = \min(Y, T)$ months. A replacement after failure costs 13 thousand AZN (emergency crew and lost output); a planned replacement at age T costs 5.5 thousand AZN.

(a) X has a mixed distribution. Find the probability mass at the discrete point, $P(X = 34)$: ____________

(b) Find the distribution function of X for $0 \leq y < 34$: $F_X(y) =$ ____________ (a formula in y; for $y \geq 34$, $F_X(y) = 1$)

(c) $E(X) =$ ____________ months

(d) The expected cost of one replacement cycle: ____________ thousand AZN

(e) Over many cycles the cost per month of operation is $E(\text{cost per cycle})/E(X)$. Find it: ____________ thousand AZN per month

(f) Find the same long-run cost per month if the operator replaced units only on failure (no planned replacement): ____________ thousand AZN per month

Give numerical answers to at least four significant figures.