Week 11, Problem Set 2 - worked solutions

Wackerly 4.9-4.11 - Other Expected Values, Tchebysheff's Theorem, and Chapter 4 in Review

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

After a motor-insurance claim is approved, the delay Y (in days) until the insurer actually pays is modelled by the exponential density

$$f(y) = \begin{cases} \frac{1}{16}\, e^{-y/16}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Use Definition 4.14 to find the moment-generating function of Y: $m(t) =$ ______________ (a formula in t)

(b) The integral in (a) is finite only for $t < b$. Find b: ______________

(c) Differentiate $m(t)$ to find $E(Y) =$ ______________ days

(d) and $V(Y) =$ ______________ (days squared).

(e) A court order makes the insurer pay penalty interest compounded continuously at 0.95% per day of delay, so a claim of 1 AZN is settled as $e^{0.0095\,Y}$ AZN. Find the expected settlement multiplier $E\left(e^{0.0095\,Y}\right) =$

____

Give numerical answers to at least four significant figures.

(a) By Definition 4.14,

$$m(t) = E(e^{tY}) = \int_0^\infty e^{ty}\,\frac{1}{16}e^{-y/16}\,dy = \frac{1}{16}\int_0^\infty \exp\left[-y\left(\frac{1}{16} - t\right)\right]dy = \frac{1}{16}\cdot\frac{1}{1/16 - t} = \frac{1}{1 - 16t}.$$

This is Example 4.13 with $\alpha = 1$, $\beta = 16$.

(b) The integral converges only when the exponent is negative, $1/16 - t > 0$, that is $t < 1/16 = 0.062500$.

(c) $m'(t) = 16(1 - 16t)^{-2}$, so $E(Y) = m'(0) = 16$.

(d) $m''(t) = 2(16)^2(1 - 16t)^{-3}$, so $E(Y^2) = m''(0) = 512$ and $V(Y) = 512 - 16^2 = 256$.

(e) $E(e^{sY})$ is the MGF evaluated at $t = s = 0.0095$, which is below b:

$$E\left(e^{0.0095Y}\right) = m(0.0095) = \frac{1}{1 - 0.1520} = 1.1792.$$

Compare the multiplier at the mean delay, $e^{0.0095\times 16} = 1.1642$: the expected multiplier is larger, because long delays are penalised exponentially and the exponential delay has a long right tail.

Problem 2

seed 20260925

A broker splits client orders into blocks. The size V of a block (in thousand shares of a Baku-listed bank) has moment-generating function

$$m(t) = (1 - 2.8\,t)^{-3}, \qquad t < 1/2.8.$$

(a) $E(V) =$ ______________

(b) $E(V^2) =$ ______________

(c) $V(V) =$ ______________

(d) The third moment about the origin, $\mu_3' = E(V^3) =$ ______________

(e) Executing a block of size V costs $C = 45\,V + 9\,V^2$ AZN: commission plus a market-impact term that grows with the square of the block. Find $E(C) =$ ______________ AZN

Give answers to at least four significant figures.

By Theorem 3.12 (valid for continuous variables, Section 4.9), $\mu_k' = m^{(k)}(0)$. With $\beta = 2.8$, $\alpha = 3$:

$$m'(t) = 3\beta(1-\beta t)^{-4}, \quad m''(t) = 3 \cdot 4\,\beta^2(1-\beta t)^{-5}, \quad m'''(t) = 3 \cdot 4 \cdot 5\,\beta^3(1-\beta t)^{-6}.$$

(a) $E(V) = m'(0) = 3 \times 2.8 = 8.4000$.

(b) $E(V^2) = m''(0) = 3 \cdot 4 \cdot 2.8^2 = 94.0800$.

(c) $V(V) = 94.0800 - 8.4000^2 = 23.5200$, which is $\alpha\beta^2$, as Theorem 4.8 says it must be: $m(t)$ is the MGF of a gamma variable with $\alpha = 3$, $\beta = 2.8$ (Example 4.13).

(d) $\mu_3' = m'''(0) = 3 \cdot 4 \cdot 5 \cdot 2.8^3 = 1317.1200$, the pattern $\alpha(\alpha+1)\cdots(\alpha+k-1)\beta^k$ of Example 4.14.

(e) As in Example 4.15, linearity of expectation (Theorem 4.5) needs only the first two moments, not the distribution of C:

$$E(C) = 45\,E(V) + 9\,E(V^2) = 45(8.4000) + 9(94.0800) = 1224.72.$$

Note $E(V^2) \neq [E(V)]^2$: pricing the impact at the average block size would understate the expected cost by $9\,V(V)$.

Problem 3

seed 20260925

A bank's stress test models three quantities for the coming quarter, all in million AZN. Their moment-generating functions are

$$m_1(t) = (1 - 2.9\,t)^{-2}, \qquad m_2(t) = \frac{1}{1 - 3.00\,t}, \qquad m_3(t) = e^{7t + 8.50\,t^2}.$$

X_1 is the operational loss, X_2 the loss on a single large exposure and X_3 the trading profit-and-loss.

(a) X_1 has a gamma distribution with $\alpha =$ ______________ and $\beta =$ ______________

(b) X_2 is exponential. Its mean is __________ and its standard deviation is __________

(c) X_3 is normal with mean $\mu =$ __________ and variance $\sigma^2 =$ __________

(d) Which quantity has the largest variance?

- ◯ The loss X_1
- ◯ The loss X_2
- ◯ The trading profit-and-loss X_3

Two random variables with the same MGF have the same distribution (Section 4.9), so it is enough to match each $m_i(t)$ with a known MGF.

(a) The gamma MGF is $(1-\beta t)^{-\alpha}$ (Example 4.13), so $\alpha = 2$, $\beta = 2.9$, and $V(X_1) = \alpha\beta^2 = 16.8200$.

(b) $1/(1-\beta t)$ is the gamma MGF with $\alpha = 1$: exponential with mean $\beta = 3.00$. An exponential variable has $\sigma = \beta$, so the standard deviation is also 3.00 and $V(X_2) = 9.0000$.

(c) The normal MGF is $e^{\mu t + \sigma^2 t^2/2}$ (Exercise 4.138). Matching the coefficient of t gives $\mu = 7$; matching the coefficient of t^2 gives $\sigma^2/2 = 8.50$, so $\sigma^2 = 17.0000$ ($\sigma = 4.1231$). A common slip is to read 8.50 itself as the variance.

(d) Comparing 16.8200, 9.0000 and 17.0000: the largest is “The trading profit-and-loss X_3”.

Problem 4

seed 20260925

(a) A mobile operator in Baku wants to advertise a range of monthly data use that excludes no more than 12% of its subscribers. All it knows is that monthly use has mean 12 GB and standard deviation 1.92 GB; the shape of the distribution is unknown. Using Tchebysheff’s theorem, the narrowest interval of the form $(\mu - k\sigma,\ \mu + k\sigma)$ that is guaranteed to do this is

from __________ GB to __________ GB.

(b) A bank’s cash-logistics team wants the daily withdrawals Y at a branch ATM to fall within 36 thousand AZN of their mean μ on at least 82% of days. Again only μ and σ are known. What is the largest standard deviation σ for which Tchebysheff’s theorem still guarantees this?

$\sigma_{\max} =$ __________ thousand AZN

Give answers to at least four significant figures.

Theorem 4.13: for any $k > 0$, $P(|Y-\mu| \geq k\sigma) \leq 1/k^2$, whatever the distribution of Y.

(a) Excluding at most 0.12 requires $1/k^2 = 0.12$, so $k = 1/\sqrt{0.12} = 2.8868$. The interval is

$$12 \pm 2.8868 \times 1.92 = (6.4574,\ 17.5426).$$

Any smaller k would give a bound above 0.12 and so no guarantee.

(b) Write $36 = k\sigma$. The guarantee $P(|Y-\mu| < k\sigma) \geq 1 - 1/k^2 \geq 0.82$ needs $1/k^2 \leq 0.18$, i.e. $k \geq 1/\sqrt{0.18}$. Since $k = 36/\sigma$, this is

$$\sigma \leq 36\sqrt{0.18} = 15.2735.$$

These are worst-case guarantees; if the distribution were known (normal, say) a larger σ would usually be acceptable.

Problem 5

seed 20260925

The daily return Y (in percent) of a stock index is modelled as normal with mean $\mu = 0.03$ and standard deviation $\sigma = 1.9$. The risk desk flags a day as a "large move" when the return is at or below -3.58 or at or above 3.64.

(a) A large move is a deviation of at least $k\sigma$ from the mean. Find $k =$ ____________

(b) Suppose the desk did not trust the normal model and knew only μ and σ. Give Tchebysheff's upper bound for the probability of a large move: ____________ (at least four significant figures)

(c) Under the normal model, find the exact probability of a large move, using Table 4, to four decimal places:

____

(a) Both thresholds are 3.61 from $\mu = 0.03$, so $k = 3.61/1.9 = 1.90$.

(b) Theorem 4.13: $P(|Y-\mu| \geq k\sigma) \leq 1/k^2 = 1/1.90^2 = 0.2770$.

(c) With $Z = (Y-\mu)/\sigma$ standard normal,

$$P(|Y-\mu| \geq 1.90\sigma) = P(Z \leq -1.90) + P(Z \geq 1.90) = 2A(1.90) = 2(0.0287) = 0.0574,$$

where $A(z)$ is the Table 4 tail area to the right of z.

The bound is about 4.8 times the normal probability. Tchebysheff's theorem must hold for every distribution with this mean and variance, including very heavy-tailed ones, so it is necessarily conservative for the normal. That is also why a desk that doubts normality (fat-tailed returns) might prefer it.

Problem 6

seed 20260925

A company holds a revolving credit line with a limit of $\theta = 19$ million AZN. The amount Y drawn at the end of a quarter (million AZN) has density

$$f(y) = \begin{cases} \dfrac{2\,y}{19^2}, & 0 \leq y \leq 19, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) $\mu_1' = E(Y) =$ ____________

(b) $\mu_2' = E(Y^2) =$ ____________

(c) the second central moment $\mu_2 = V(Y) =$ ____________

(d) For the quarter the bank earns interest of 11% on the drawn amount and a commitment fee of 1% on the undrawn amount $19 - Y$. Find the bank's expected income from this line, in million AZN: ______________

Give answers to at least four significant figures.

By Definition 4.13, for $k = 1, 2, \ldots$

$$\mu'_k = E(Y^k) = \int_0^{19} y^k \, \frac{2\, y^1}{19^2} \, dy = \frac{2}{19^2} \cdot \frac{19^{k+2}}{k+2} = \frac{2\,(19)^k}{k+2}.$$

Replacing the power 1 by 0 gives the uniform density and Example 4.12, $\theta^k/(k+1)$.

(a) $k = 1$: $\mu'_1 = 2(19)/3 = 12.6667$.

(b) $k = 2$: $\mu'_2 = 2(19)^2/4 = 180.5000$.

(c) $\mu_2 = \mu'_2 - (\mu'_1)^2 = 180.5000 - 12.6667^2 = 20.0556$.

(d) Income is $0.110\,Y + 0.0100\,(19 - Y)$, a linear function of Y, so

$$E(\text{income}) = 0.110(12.6667) + 0.0100(6.3333) = 1.45667.$$

Problem 7

seed 20260925

The overnight interbank rate R (percent) for a given day is modelled as uniformly distributed on the band $(4.75,\ 7.75)$. A bank that must borrow overnight has a funding cost of

$$C = 11 + 45\,R \quad \text{thousand AZN.}$$

(a) From Definition 4.14, find the moment-generating function of R for $t \neq 0$: $m_R(t) =$ ______________

(b) Use $m_{aY+b}(t) = e^{tb}\, m_Y(at)$ (Exercise 4.137) to find the MGF of the cost, for $t \neq 0$: $m_C(t) =$

____

(c) The MGF in (b) identifies the distribution of C: it is uniform on an interval (c_1, c_2) with $c_1 =$ ______________ and $c_2 =$ ______________

(d) $V(C) =$ ______________

Enter formulas in t, for example (e^(5*t)-e^(4*t))/t; give numbers to at least four significant figures.

(a) The uniform density on $(\theta_1, \theta_2) = (4.75, 7.75)$ is $1/3$ there, so for $t \neq 0$

$$m_R(t) = \int_{4.75}^{7.75} \frac{e^{tr}}{3} \, dr = \frac{e^{7.75t} - e^{4.75t}}{3\,t},$$

the case $\theta_1 = 4.75$, $\theta_2 = 7.75$ of Exercise 4.141 ($m_R(0) = 1$ by continuity).

(b) With the constant 11 and the multiplier 45,

$$m_C(t) = E\big(e^{t(11+45R)}\big) = e^{11t}\, m_R(45t) = e^{11t}\, \frac{e^{45\cdot 7.75\,t} - e^{45\cdot 4.75\,t}}{3 \cdot 45\,t} = \frac{e^{359.75t} - e^{224.75t}}{135\,t}.$$

(c) This is exactly the form in (a) with $\theta_1 = 224.75$, $\theta_2 = 359.75$ (note $359.75 - 224.75 = 135$ in the denominator). By the uniqueness of MGFs, C is uniform on $(224.75,\ 359.75)$: a linear function with positive slope maps the band onto a band.

(d) A uniform variable on an interval of length L has variance $L^2/12$ (Theorem 4.6), so $V(C) = 135^2/12 = 1518.7500$, the same as $45^2\,V(R) = 45^2(3)^2/12$.

Problem 8

seed 20260925

The daily return Y (percent) of a stock index is normal with mean $\mu = 0.1$ and standard deviation $\sigma = 2.4$, so (Exercise 4.138)

$$m_Y(t) = e^{\mu t + \sigma^2 t^2/2}.$$

A dealer is short the index and also earns a fixed carry, so its daily profit, in thousand AZN, is

$$X = 5.5 - 5\,Y.$$

(a) Find the moment-generating function of X: $m_X(t) =$ ______________ (a formula in t)

(b) $E(X) =$ ______________

(c) $V(X) =$ ______________

(d) The MGF shows that X is normal. Using Table 4, find to four decimal places the probability of a loss of 10.48 thousand AZN or worse, i.e. $P(X < -10.48)$: ______________

(a) By Exercise 4.137 with multiplier -5 and constant 5.5,

$$m_X(t) = E\left(e^{t(5.5-5Y)}\right) = e^{5.5t}\,m_Y(-5t) = e^{5.5t}\exp\left[0.1(-5t) + \tfrac{5.76}{2}(5t)^2\right] = e^{5t+72\,t^2}.$$

(b)-(c) This is the normal MGF $e^{\mu_X t + \sigma_X^2 t^2/2}$ with $\mu_X = 5.5 - 0.5 = 5$ and $\sigma_X^2/2 = 72$, so $V(X) = 144$ and $\sigma_X = 5 \times 2.4 = 12$. (Differentiating m_X at 0 gives the same.) By uniqueness of MGFs, X is normal: a linear function of a normal variable is normal, even with a negative multiplier.

(d) Standardise: $z = (-10.48 - 5)/12 = -1.29$, so

$$P(X < -10.48) = P(Z < -1.29) = A(1.29) = 0.0985.$$

Problem 9

seed 20260925

A property insurer in Baku reinsures a portfolio abroad. The size Y of a claim, in thousand USD, is gamma distributed with $\alpha = 2$ and $\beta = 4.7$. Each claim is converted to manat at the fixed rate of 1.70 AZN per USD, and a handling fee of 0.7 thousand AZN is added, so the cost of a claim in thousand AZN is

$$C = 0.7 + 1.7\,Y.$$

(a) Find the moment-generating function of C: $m_C(t) =$ ______________ (a formula in t)

(b) $m_C(t)$ exists for $t < b$. Find b: ______________

(c) The converted claim $1.7\,Y$ (without the fee) is gamma distributed with $\alpha =$ ______________ and $\beta =$ ______________

(d) Use $m_C(t)$ to find $E(C) =$ ______________

(e) and $V(C) =$ ______________

Give numerical answers to at least four significant figures.

(a) From Example 4.13, $m_Y(t) = (1 - 4.7t)^{-2}$ for $t < 1/4.7$. By Exercise 4.137,

$$m_C(t) = E\left(e^{t(0.7+1.7Y)}\right) = e^{0.7t}\, m_Y(1.7t) = e^{0.7t}\,(1 - 7.99\,t)^{-2}.$$

(b) $m_Y(1.7t)$ needs $1.7t < 1/4.7$, i.e. $t < 1/7.99 = 0.125156$.

(c) The MGF of $1.7Y$ is $m_Y(1.7t) = (1 - 7.99t)^{-2}$, the gamma MGF with $\alpha = 2$ and $\beta = 1.7 \times 4.7 = 7.99$. By uniqueness, rescaling a gamma variable keeps it gamma, with the same shape and the scale multiplied: a change of currency does not change the shape of the claim distribution. (Adding the fee shifts it, so C itself is not gamma.)

(d) Write $m_C(t) = e^{0.7t} g(t)$, $g(t) = (1 - 7.99t)^{-2}$, with $g(0) = 1$, $g'(0) = 2(7.99) = 15.9800$ and $g''(0) = 2 \cdot 3\,(7.99)^2 = 383.0406$. Then $m_C'(0) = 0.7 + 15.9800 = 16.6800$.

(e) $m_C''(0) = 0.7^2 + 2(0.7)(15.9800) + 383.0406$, and subtracting $[m_C'(0)]^2 = (0.7 + 15.9800)^2$ leaves

$$V(C) = 383.0406 - 15.9800^2 = 2(7.99)^2 = 127.6802,$$

as it must be: the fee shifts C without changing its spread, and $V(C) = 1.7^2\, V(Y)$.

Problem 10

seed 20260925

An insurer's risk team simulates the time W (in days) between successive motor claims by drawing U uniformly on $(0, 1)$ and setting

$$W = g(U) = -2.0\, \ln U.$$

(a) Use Theorem 4.12, $E\left[e^{tg(U)}\right] = \int_0^1 e^{tg(u)}\, f_U(u)\, du$, to find the moment-generating function of W: $m_W(t) =$ ______________ (a formula in t)

(b) The integral converges only for $t < b$. Find b: ______________

(c) What is the distribution of W?

- ◯ Exponential with mean 2.0
- ◯ Gamma with alpha = 2 and beta = 2.0
- ◯ Normal with mean 2.0 and variance 1
- ◯ Uniform on (0, 2.0)

(d) Find $P(W > 5.60)$: ______________

(e) Find the median of W, the value $\phi_{0.5}$ with $P(W \le \phi_{0.5}) = 0.5$: ______________ days

Give numerical answers to at least four significant figures.

(a) U has density 1 on $(0,1)$, and $e^{tg(u)} = e^{-2.0t\ln u} = u^{-2.0t}$. By Theorem 4.12,

$$m_W(t) = \int_0^1 u^{-2.0t}\,du = \left[\frac{u^{1-2.0t}}{1-2.0t}\right]_0^1 = \frac{1}{1-2.0t}.$$

(b) The antiderivative vanishes at $u = 0$ only when the exponent $1 - 2.0t$ is positive, so the integral is finite for $t < 1/2.0 = 0.500000$.

(c) $1/(1-\beta t)$ is the MGF of the exponential distribution with mean β (Example 4.13 with $\alpha = 1$). By the uniqueness of MGFs (Section 4.9), W is exponential with mean 2.0 days. This is the standard way simulation software turns uniform random numbers into exponential waiting times.

(d) $P(W > 5.60) = e^{-5.60/2.0} = e^{-2.8000} = 0.0608$.

(e) Solve $1 - e^{-\phi/2.0} = 0.5$: $\phi_{0.5} = 2.0\ln 2 = 1.3863$ days, below the mean 2.0 because the exponential is right-skewed.

Problem 11

seed 20260925

The time Y (minutes) that an operator at a bank's call centre spends on a customer call is exponential with mean $\beta = 4.0$. Staffing is planned around the interval $\mu \pm k\sigma$ with $k = 2.85$.

(a) Find the upper end $\mu + k\sigma$ of the interval: ______________ minutes

(b) Find the exact probability $P(|Y-\mu| < k\sigma)$: ______________

(c) What lower bound does Tchebysheff's theorem give for the probability in (b)? ______________

(d) Find the exact probability that a call runs past the upper end of the interval, $P(Y \geq \mu + k\sigma)$: ______________

Give answers to at least four significant figures.

For the exponential, $\mu = \sigma = \beta = 4.0$ (Theorem 4.10).

(a) $\mu + k\sigma = 4.0(1 + 2.85) = 15.4000$.

(b) The lower end is $4.0(1 - 2.85) = -7.4000 < 0$, below the support, so only the upper end matters:

$$P(|Y-\mu| < k\sigma) = P(Y < 15.4000) = 1 - e^{-15.4000/4.0} = 1 - e^{-3.85} = 0.9787.$$

Notice that the answer does not depend on β: in units of β every exponential looks the same.

(c) Theorem 4.13: $P(|Y-\mu| < k\sigma) \geq 1 - 1/k^2 = 1 - 1/2.85^2 = 0.8769$. The true value 0.9787 is well above the bound; the theorem is correct but conservative.

(d) $P(Y \geq 15.4000) = e^{-3.85} = 0.0213$. Here the whole probability outside the interval sits in the right tail. Tchebysheff's bound on the two tails together is $1/k^2 = 0.1231$, several times larger.

Problem 12

seed 20260925

At a Baku bank, the time Y (hours of officer work) needed to process a small-business loan application has, from past experience, a gamma distribution with $\alpha = 2.8$ and $\beta = 2.8$. A newly hired officer takes 22.1 hours on an application.

(a) $\mu =$ ______________ hours and $\sigma =$ ______________ hours

(b) By how many standard deviations does 22.1 exceed the mean? $k =$ ______________

(c) Tchebysheff's upper bound for $P(|Y - \mu| \geq k\sigma)$ with this k: ______________

(d) The bank's quality-control rule flags a new officer for review when, under the past distribution, the probability of a deviation at least as large as the one observed is guaranteed to be at most 0.05. What does the rule say here?

- ○ Flag: a deviation this large has probability at most 0.05 under past experience
- ○ Do not flag: Tchebysheff's bound exceeds 0.05, so chance cannot be ruled out

Give numerical answers to at least four significant figures.

(a) Theorem 4.8: $\mu = \alpha\beta = 2.8 \times 2.8 = 7.8400$ and $\sigma^2 = \alpha\beta^2 = 21.9520$, so $\sigma = 4.6853$.

(b) $22.1 - 7.8400 = 14.2600$, and $k = 14.2600/4.6853 = 3.0436$.

(c) Theorem 4.13: $P(|Y - \mu| \geq 3.0436\sigma) \leq 1/3.0436^2 = 0.1080$.

(d) The bound is above 0.05, so the rule does not flag the officer. As in Example 4.17, a small bound means that either a rare event has happened or, more plausibly, the new officer is slower than those before; a bound above 0.05 does not show the event is common, only that Tchebysheff's theorem, using nothing but μ and σ, cannot rule chance out. The exact gamma tail would be smaller than the bound.

Problem 13

seed 20260925

A filling station on the Baku-Guba highway sells a random amount Y of diesel each day, in thousand litres, with density

$$f(y) = \begin{cases} \frac{3y^2}{3^3}, & 0 \leq y \leq 3, \\ 0, & \text{elsewhere.} \end{cases}$$

The station's margin is 85 AZN per thousand litres on days when $Y \leq 2.4$. On days when $Y > 2.4$ the supplier's volume rebate raises the margin to 140 AZN per thousand litres on the whole day's volume.

(a) Find $P(Y > 2.4)$: ______________

(b) Find the station's expected daily profit: ______________ AZN

(c) How much of the expected daily profit in (b) is due to the rebate, that is, the expected profit minus what it would be if the margin were always 85? ______________ AZN

Give answers to at least four significant figures.

The daily profit is the discontinuous function

$$g(Y) = \begin{cases} 85\,Y, & 0 \le Y \le 2.4, \\ 140\,Y, & 2.4 < Y \le 3. \end{cases}$$

(a) $P(Y > 2.4) = \int_{2.4}^{3} 3y^2/3^3\,dy = 1 - (2.4/3)^3 = 0.4880$.

(b) By Theorem 4.4, split the integral at the jump, as in Example 4.18:

$$E[g(Y)] = 85 \int_0^{2.4} y\,\frac{3y^2}{27.000}\,dy + 140 \int_{2.4}^{3} y\,\frac{3y^2}{27.000}\,dy.$$

Since $\int y \cdot 3y^2\,dy = 3y^4/4$,

$$\int_0^{2.4} \frac{3y^3}{27.000} dy = \frac{3(33.1776)}{4(27.000)} = 0.9216, \qquad \int_{2.4}^{3} \frac{3y^3}{27.000} dy = \frac{3(81.0000 - 33.1776)}{4(27.000)} = 1.3284,$$

so $E[g(Y)] = 85(0.9216) + 140(1.3284) = 264.31$ AZN.

(c) With a constant margin the expected profit would be $85\,E(Y) = 85 \times 3(3)/4 = 85(2.2500)$. The difference is $(140 - 85)\int_{2.4}^{3} y f(y)\,dy = 55(1.3284) = 73.06$ AZN. Note that the rebate is worth 55 times $E[Y\,I(Y > 2.4)]$, not 55 times $E(Y)P(Y > 2.4)$: the high-volume days carry more litres.

Problem 14

seed 20260925

An importer's containers arriving at the Port of Baku take a customs-clearance time Y (days) that is uniformly distributed on $(1,\ 8.5)$. The terminal charges a flat 375 AZN per container for any clearance time up to 3.25 days. Beyond 3.25 days it charges 375 AZN plus 70 AZN per additional day, prorated: a clearance time of $3.25 + 1.5$ days costs $375 + 70(1.5)$ AZN.

(a) The probability that the importer pays more than the flat charge: ______________

(b) The expected cost per container: ______________ AZN

(c) The expected cost per container among containers that are not cleared within 3.25 days: ______________ AZN

Give answers to at least four significant figures.

The cost is $g(Y) = 375 + 70\,(Y - 3.25)^+$, where $(u)^+ = \max(u, 0)$: flat up to 3.25, then rising with slope 70. The density is $1/7.5$ on $(1, 8.5)$.

(a) $P(Y > 3.25) = (8.5 - 3.25)/7.5 = 0.7000$.

(b) By Theorem 4.4, only the part of the range above 3.25 contributes to the extra charge:

$$E[(Y - 3.25)^+] = \int_{3.25}^{8.5} (y - 3.25)\,\frac{1}{7.5}\,dy = \frac{(5.25)^2}{2(7.5)} = 1.8375,$$

so $E[g(Y)] = 375 + 70(1.8375) = 503.62$ AZN.

(c) Given $Y > 3.25$, Y is uniform on $(3.25, 8.5)$, so the expected excess is $5.25/2 = 2.625$ days and the conditional expected cost is $375 + 70(2.625) = 558.75$ AZN. Consistently, $375 + 70\,P(Y > 3.25)\,E(Y - 3.25 \mid Y > 3.25)$ reproduces (b).

Problem 15

seed 20260925

The damage Y (thousand AZN) in a motor accident reported to an insurer is exponential with mean $\beta = 1.8$. A policy has a deductible of $d = 1.4$: the insurer pays nothing when $Y \le d$ and $Y - d$ otherwise.

(a) The probability that a reported accident leads to a payment: ______________

(b) The expected payment per reported accident: ______________ thousand AZN

(c) The policy also has a limit: losses above $u = 11$ are covered only up to u, so the payment is $\min\{(Y-d)^+,\ u-d\}$, at most 9.6. Find the expected payment per reported accident now: ______________ thousand AZN

(d) With the deductible and the limit, the expected payment per accident that does lead to a payment: ______________ thousand AZN

Give answers to at least four significant figures.

The density is $f(y) = e^{-y/1.8}/1.8$, $y > 0$, and $P(Y > y) = e^{-y/1.8}$.

(a) $P(Y > d) = e^{-1.4/1.8} = 0.4594$.

(b) By Theorem 4.4 the integral runs over $y > d$ only:

$$E[(Y-d)^+] = \int_d^\infty (y-d)\,\frac{e^{-y/\beta}}{\beta}\,dy = e^{-d/\beta}\int_0^\infty v\,\frac{e^{-v/\beta}}{\beta}\,dv = \beta e^{-d/\beta} = 1.8(0.4594) = 0.8270,$$

substituting $v = y - d$.

(c) Now the payment function has three pieces: 0 for $y \le d$, $y - d$ for $d < y \le u$, and $u - d$ for $y > u$. So

$$E = \int_d^u (y-d)f(y)\,dy + (u-d)P(Y > u).$$

Integrating by parts, $\int_d^u (y-d)f(y)\,dy = -(u-d)e^{-u/\beta} + \beta(e^{-d/\beta} - e^{-u/\beta})$; the first term cancels $(u-d)P(Y>u)$ and

$$E = \beta\big(e^{-d/\beta} - e^{-u/\beta}\big) = 1.8(0.4594 - 0.002218) = 0.8230.$$

The limit removes only $\beta e^{-u/\beta}$ from (b), little when u is several means above d.

(d) Divide by the probability of a payment: $0.8230/0.4594 = 1.7913$. Without the limit this would be exactly $\beta = 1.8$ by the memoryless property: the excess over d of an exponential loss that exceeds d is again exponential with mean β.

Problem 16

seed 20260925

For a home-insurance policy in Baku, the total amount Y (thousand AZN) the insurer pays in a year is 0 with probability 0.75, since most households make no claim. Given that there is a claim, the amount paid has the exponential density

$$f(y) = \frac{1}{7.5}e^{-y/7.5}, \qquad y > 0.$$

(a) Find the distribution function of Y at 4.5: $F(4.5) =$ ______________

(b) Find $P(Y > 4.5)$: ______________

(c) Find $E(Y)$, the pure premium for the policy: ______________ thousand AZN

(d) Find $V(Y)$: ______________

Give answers to at least four significant figures.

Y has a mixed distribution (Section 4.11): $F(y) = c_1F_1(y) + c_2F_2(y)$ with $c_1 = 0.75$, F_1 the step function of a variable $X_1 \equiv 0$, $c_2 = 0.25$, and $F_2(y) = 1 - e^{-y/7.5}$ for $y \geq 0$, the distribution function of an exponential X_2 with mean 7.5.

(a) For $y \geq 0$, $F(y) = 0.75 + 0.25(1 - e^{-y/7.5})$, which jumps by 0.75 at 0 as in Figure 4.18. At $y = 4.5$: $e^{-4.5/7.5} = 0.5488$, so $F(4.5) = 0.75 + 0.25(1 - 0.5488) = 0.8628$.

(b) $P(Y > 4.5) = 1 - F(4.5) = 0.25\, e^{-4.5/7.5} = 0.1372$, as in Example 4.19.

(c) Definition 4.15 with $g(y) = y$: $E(Y) = 0.75\, E(X_1) + 0.25\, E(X_2) = 0.75(0) + 0.25(7.5) = 1.8750$.

(d) With $g(y) = y^2$ and $E(X_2^2) = 2\beta^2 = 112.50$: $E(Y^2) = 0.25(112.50) = 28.1250$, so $V(Y) = 28.1250 - 1.8750^2 = 24.6094$.

The point mass at zero lowers the mean much more than the spread: the coefficient of variation σ/μ is much larger than the 1 of a pure exponential, which is why the insurer needs many policies to price reliably.

Problem 17

seed 20260925

A trader holds a European call option on a share listed on the Baku Stock Exchange with strike price $K = 30.275$ AZN. Her model for the share price S at expiry is normal with mean $\mu = 32$ AZN and standard deviation $\sigma = 11.5$ AZN. The call pays $(S - K)^+ = \max(S - K,\, 0)$ at expiry.

(a) Using Table 4, the probability that the call finishes in the money, $P(S > K)$, to four decimal places:

____

(b) The expected payoff of the call, $E[(S - K)^+]$: ______________ AZN

(c) The expected payoff of the put with the same strike, which pays $(K - S)^+$: ______________ AZN

Hint: write $S = \mu + \sigma Z$ with Z standard normal, and use $\int_z^\infty u\,\phi(u)\,du = \phi(z)$, where $\phi(u) = e^{-u^2/2}/\sqrt{2\pi}$. Give (b) and (c) to at least four significant figures.

(a) The strike is $z = (K - \mu)/\sigma = (30.275 - 32)/11.5 = -0.15$ standard deviations from the mean, so $P(S > K) = P(Z > -0.15) = 0.5596$.

(b) The payoff $g(S) = (S - K)^+$ is 0 below K and $S - K$ above it, so by Theorem 4.4 the integral runs over $s > K$ only. Substituting $s = \mu + \sigma u$, $s - K = \sigma(u - z)$:

$$E[(S-K)^+] = \int_K^\infty (s-K)f(s)\,ds = \sigma\int_z^\infty (u-z)\phi(u)\,du = \sigma\big[\phi(z) - z\,P(Z>z)\big].$$

Here $\phi(-0.15) = 0.3945$ and $z\,P(Z>z) = -0.0839$, so

$$E[(S-K)^+] = 11.5\,(0.3945 + 0.0839) = 5.5019.$$

(c) $(S-K)^+ - (K-S)^+ = S-K$ for every S, so taking expectations (put-call parity for expected payoffs)

$$E[(K-S)^+] = E[(S-K)^+] - (\mu - K) = 5.5019 - (1.725) = 3.7769.$$

Directly: \(\sigma[\phi(z) + z\,P(Z

The expected payoff is not $(\mu-K)^+$: the option's value comes from the spread σ, which is why option prices rise with volatility.

Problem 18

seed 20260925

A business-interruption policy sold to restaurants in Baku pays for income lost while a restaurant is closed. The annual loss Y (in ten thousand AZN) is 0 with probability 0.66. Given a loss, it has the gamma density with $\alpha = 2$ and $\beta = 4.7$:

$$f(y) = \frac{y\,e^{-y/4.7}}{4.7^2}, \qquad y > 0.$$

(a) $E(Y) =$ _______________

(b) $V(Y) =$ _______________

(c) The policy has a deductible of $d = 2.0$, so it pays $(Y-d)^+$. Find the probability that the policy pays anything in a year: _______________

(d) Find the expected annual payment $E[(Y-d)^+]$: _______________

(e) Find the expected payment in a year in which the policy does pay: _______________

Give answers to at least four significant figures.

Write the distribution as the mixture of Definition 4.15: with probability $c_1 = 0.66$ the loss is $X_1 \equiv 0$, and with probability $c_2 = 0.34$ it is X_2, gamma with $\alpha = 2$, $\beta = 4.7$. For any g, $E[g(Y)] = 0.66\,g(0) + 0.34\,E[g(X_2)]$.

(a) Theorem 4.8: $E(X_2) = 2\beta = 9.4$, so $E(Y) = 0.34(9.4) = 3.1960$.

(b) From Example 4.14, $E(X_2^2) = \alpha(\alpha+1)\beta^2 = 6\beta^2 = 132.54$, so $E(Y^2) = 0.34(132.54) = 45.0636$ and $V(Y) = 45.0636 - 3.1960^2 = 34.8492$.

(c) A payment requires a loss above d. Integrating by parts,

$$P(X_2 > d) = \int_d^\infty \frac{ye^{-y/\beta}}{\beta^2}dy = e^{-d/\beta}\left(1 + \frac{d}{\beta}\right) = 0.6534(1 + 0.4255) = 0.9315,$$

the Poisson-sum form for $\alpha = 2$. So $P(Y > d) = 0.34(0.9315) = 0.3167$.

(d) $g(y) = (y-d)^+$ has $g(0) = 0$, so $E[(Y-d)^+] = 0.34\,E[(X_2-d)^+]$. Using $E[(X-d)^+] = \int_d^\infty P(X>y)\,dy$ (or integrating $(y-d)f(y)$ directly),

$$E[(X_2-d)^+] = \int_d^\infty e^{-y/\beta}\Big(1+\frac{y}{\beta}\Big)dy = \beta e^{-d/\beta} + (d+\beta)e^{-d/\beta} = e^{-d/\beta}(2\beta+d) = 0.6534(11.4) = 7.4490.$$

Hence $E[(Y-d)^+] = 0.34(7.4490) = 2.5327$.

(e) $E[(Y-d)^+ \mid Y > d] = 2.5327/0.3167 = 7.9970$. The zero-loss restaurants cancel out of this ratio: p_0 affects how often the insurer pays, not how much it pays when it does.

Problem 19

seed 20260925

A bank models its annual operational-risk loss X (million AZN) as gamma with $\alpha = 2$ and $\beta = 3.8$, so $m_X(t) = (1-3.8t)^{-2}$. The regulator asks how likely a loss of $x_0 = 22.65$ million AZN or more is. A risk analyst rescales the loss to

$$U = \frac{2X}{\beta} = \frac{2X}{3.8}.$$

(a) Find the moment-generating function of U: $m_U(t) =$ ______________ (a formula in t)

(b) The MGF shows that U has a χ^2 distribution. With how many degrees of freedom? $\nu =$ ______________

(c) Using only the mean and variance of X, give Tchebysheff's upper bound for $P(X \geq 22.65)$, through $P(|X-\mu| \geq 22.65 - \mu)$: ______________

(d) Find the exact probability $P(X \geq 22.65)$: ______________

Give numerical answers to at least four significant figures.

(a) $U = aX$ with $a = 2/3.8$, so by Exercise 4.137

$$m_U(t) = m_X\Big(\frac{2t}{3.8}\Big) = \Big(1 - 3.8\cdot\frac{2t}{3.8}\Big)^{-2} = (1-2t)^{-2}, \quad t < 1/2.$$

The scale β has disappeared.

(b) $(1-2t)^{-\nu/2}$ is the MGF of a χ^2 variable with ν degrees of freedom (gamma with $\alpha = \nu/2$, $\beta = 2$, Definition 4.10). By uniqueness $\nu/2 = 2$, so $\nu = 4$. The event $X \geq 22.65$ is $U \geq 2(22.65)/3.8 = 11.9211$, which could also be looked up in a χ^2_4 table or software.

(c) $\mu = \alpha\beta = 7.6000$, $\sigma^2 = \alpha\beta^2 = 28.8800$. Since $22.65 > \mu$, $P(X \geq 22.65) \leq P(|X-\mu| \geq 15.0500)$, and Theorem 4.13 with $k\sigma = 15.0500$ gives

$$P(X \geq 22.65) \leq \frac{\sigma^2}{(15.0500)^2} = \frac{28.8800}{(15.0500)^2} = 0.1275.$$

(The same bound comes out of U, whose mean 4 and variance 2ν are μ and σ^2 rescaled.)

(d) For integer α, the gamma tail equals a Poisson sum (Exercise 4.99): $P(X \geq x_0) = \sum_{j=0}^{\alpha-1} e^{-\lambda}\lambda^j/j!$ with $\lambda = x_0/\beta = 22.65/3.8 = 5.9605$. Summing $j = 0, \ldots, 1$:

$$P(X \geq 22.65) = 0.01795.$$

Tchebysheff's bound 0.1275 is valid but several times too large here; it also ignores that the deviation is in the long right tail only.

Problem 20

seed 20260925

The pitch-control unit of a turbine at a wind farm on the Absheron peninsula has a life length Y (months) that is exponential with mean 21. The operator replaces the unit when it fails or when it reaches age $T = 34$ months, whichever comes first, so the unit is in use for $X = \min(Y, T)$ months. A replacement after failure costs 13 thousand AZN (emergency crew and lost output); a planned replacement at age T costs 5.5 thousand AZN.

(a) X has a mixed distribution. Find the probability mass at the discrete point, $P(X = 34)$: ______________

(b) Find the distribution function of X for $0 \le y < 34$: $F_X(y) =$ ______________ (a formula in y; for $y \ge 34$, $F_X(y) = 1$)

(c) $E(X) =$ ______________ months

(d) The expected cost of one replacement cycle: ______________ thousand AZN

(e) Over many cycles the cost per month of operation is $E(\text{cost per cycle})/E(X)$. Find it: ______________ thousand AZN per month

(f) Find the same long-run cost per month if the operator replaced units only on failure (no planned replacement): ______________ thousand AZN per month

Give numerical answers to at least four significant figures.

(a) $X = T$ exactly when the unit survives to age T: $P(X = 34) = P(Y > 34) = e^{-34/21} = 0.1981$.

(b) For $0 \le y < 34$, $X \le y$ happens only when $Y \le y$, so $F_X(y) = 1 - e^{-y/21}$; at $y = 34$ the distribution function jumps by 0.1981 to 1. In the notation of Section 4.11, $F_X = c_1 F_1 + c_2 F_2$ with $c_1 = 0.1981$ at the point 34, and $c_2 = 0.8019$ times the exponential distribution truncated to $(0, 34)$.

(c) Split the expectation at T (Theorem 4.4 plus the point mass):

$$E(X) = \int_0^{34} y\,\frac{e^{-y/21}}{21}\,dy + 34\,e^{-34/21} = 21\big(1 - e^{-34/21}\big) = 21(0.8019) = 16.8402,$$

after integrating by parts (the $-Te^{-T/\beta}$ boundary term cancels the point-mass term).

(d) A cycle ends in failure with probability 0.8019 and in a planned replacement with probability 0.1981: $13(0.8019) + 5.5(0.1981) = 11.5143$.

(e) $11.5143/16.8402 = 0.68374$ thousand AZN per month.

(f) With no planned replacement every cycle costs 13 and lasts $E(Y) = 21$ months on average: $13/21 = 0.61905$.

The planned policy costs 0.06470 more per month. That is no accident of the numbers: an exponential unit does not wear out (the memoryless property, Example 4.10), so a unit of age T is as good as a new one and replacing it only adds the cost 5.5. Indeed $(e) = c_1/\beta + c_2 e^{-T/\beta}/E(X) > c_1/\beta$ for every T. Age replacement pays only for components that age, such as a gamma life with $\alpha > 1$.

Answer key

Problem	Answers
1	(a) 1/(1-16*t), (b) 0.0625, (c) 16, (d) 256, (e) 1.17925
2	(a) 8.4, (b) 94.08, (c) 23.52, (d) 1317.12, (e) 1224.72
3	(a) 2, (b) 2.9, (c) 3, (d) 3, (e) 7, (f) 17, (g) Choice 3
4	(a) 6.45744, (b) 17.5426, (c) 15.2735
5	(a) 1.9, (b) 0.277008, (c) 0.0574332
6	(a) 12.6667, (b) 180.5, (c) 20.0556, (d) 1.45667
7	(a) [e^(7.75*t)-e^(4.75*t)]/(3*t), (b) [e^(359.75*t)-e^(224.75*t)]/(135*t), (c) 224.75, (d) 359.75, (e) 1518.75
8	(a) e^(5*t+72*t^2), (b) 5, (c) 144, (d) 0.0985252
9	(a) e^(0.7*t)*(1-7.99*t)^(-2), (b) 0.125156, (c) 2, (d) 7.99, (e) 16.68, (f) 127.68
10	(a) 1/(1-2*t), (b) 0.5, (c) Choice 1, (d) 0.0608101, (e) 1.38629
11	(a) 15.4, (b) 0.97872, (c) 0.876885, (d) 0.0212797
12	(a) 7.84, (b) 4.6853, (c) 3.04356, (d) 0.107953, (e) Choice 2
13	(a) 0.488, (b) 264.312, (c) 73.062
14	(a) 0.7, (b) 503.625, (c) 558.75
15	(a) 0.459426, (b) 0.826966, (c) 0.822974, (d) 1.79131
16	(a) 0.862797, (b) 0.137203, (c) 1.875, (d) 24.6094
17	(a) 0.559618, (b) 5.50185, (c) 3.77685
18	(a) 3.196, (b) 34.8492, (c) 0.316701, (d) 2.53266, (e) 7.99701
19	(a) (1-2*t)^(-2), (b) 4, (c) 0.127504, (d) 0.0179481
20	(a) 0.198087, (b) 1-e^(-y/21), (c) 16.8402, (d) 11.5143, (e) 0.683743, (f) 0.619048