Week 12, Problem Set 1

Wackerly 5.1-5.2 - Bivariate and Multivariate Probability Distributions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku bank has lent to two construction companies. Over the coming year let

- $Y_1 = 1$ if the first company defaults and $Y_1 = 0$ if it does not,
- $Y_2 = 1$ if the second company defaults and $Y_2 = 0$ if it does not.

The bank's credit-risk unit gives the joint probability function $p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2)$ below, but the entry for $(0, 0)$ has been left as an unknown constant c.

	$y_1 = 0$	$y_1 = 1$
$y_2 = 0$	c	0.13
$y_2 = 1$	0.07	0.02

Give every probability to at least four significant figures.

(a) The value of c that makes $p(y_1, y_2)$ a valid joint probability function: $c =$ _____

(b) The probability that at least one of the two companies defaults: _____

(c) The probability that exactly one of the two companies defaults: _____

Problem 2

seed 20260925

An insurer writes motor policies through its Ganja and Sumgait branches. On a given day let Y_1 be the number of large claims (0, 1 or 2) reported at Ganja and Y_2 the number reported at Sumgait. The actuaries' joint probability function $p(y_1, y_2)$ is below; the entry for $(0, 0)$ is an unknown constant c.

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	c	0.08	0.05
$y_2 = 1$	0.06	0.09	0.04
$y_2 = 2$	0.05	0.02	0.03

Give every probability to at least four significant figures.

(a) $c =$ _____

(b) $P(1 \le Y_1 \le 2,\ Y_2 \le 1) =$ _____

(c) The probability that both branches report the same number of large claims, $P(Y_1 = Y_2) =$ _____

Problem 3

seed 20260925

A customer holds two credit cards issued by different Baku banks. Over a quarter let Y_1 be the number of late payments (0, 1 or 2) on card A and Y_2 the number (0 or 1) on card B. Their joint probability function is

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	0.57	0.19	0.08
$y_2 = 1$	0.07	0.06	0.03

Let $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$ be the joint distribution function (Definition 5.2). Give every value to at least four significant figures.

(a) $F(1, 0) =$ _____

(b) $F(0, 1) =$ _____

(c) $F(1.7, 4) =$ _____

(d) $F(2, 0.5) =$ _____

Problem 4

seed 20260925

A bank's treasury desk expects two incoming wire transfers after the market opens. Let Y_1 be the arrival time of the first transfer and Y_2 that of the second, both in minutes after opening. The desk models the pair with the joint density

$$f(y_1, y_2) = \begin{cases} k, & 0 \le y_1 \le 20,\ 0 \le y_2 \le 40, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures (a fraction such as $1/500$ is also accepted).

(a) The value of k that makes f a joint density: $k =$ _____

(b) $F(8, 32) = P(Y_1 \le 8,\ Y_2 \le 32) =$ _____

(c) $P(4 \le Y_1 \le 18,\ Y_2 > 32) =$ _____

Problem 5

seed 20260925

A corporate client has two revolving credit lines with a bank. Let Y_1 and Y_2 be the proportions of the two limits in use at month end. The bank models them with the joint density

$$f(y_1, y_2) = \begin{cases} k\, y_1 y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ _____

(b) $F(0.25, 0.90) = P(Y_1 \le 0.25,\ Y_2 \le 0.90) =$ _____

(c) $P(Y_1 \le 0.25,\ Y_2 > 0.90) =$ _____

(d) The probability that more than 0.4 of the first line is in use, $P(Y_1 > 0.4) =$ _____

Problem 6

seed 20260925

A microfinance lender approves small-business loans with a ceiling of 20 thousand AZN. For a randomly chosen loan let Y_1 be the amount disbursed and Y_2 the amount repaid by the end of the first year, both in thousand AZN. Nothing can be repaid that was not disbursed, so $Y_2 \le Y_1$. The lender models the pair as uniform on the triangle this allows:

$$f(y_1, y_2) = \begin{cases} k, & 0 \le y_2 \le y_1 \le 20, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ _____

(b) The probability that at most 18 thousand AZN is disbursed, $P(Y_1 \le 18) =$ _____

(c) The probability that more than 8 thousand AZN is repaid, $P(Y_2 > 8) =$ _____

Problem 7

seed 20260925

An analyst models next month's returns on two assets traded in Baku: Y_1, the return on a bank share, and Y_2, the return on a government bond fund, both in percent. On a coarse grid each return is -2, 1 or 4, and the joint probability function is below. The entry for $(1, 1)$ is an unknown constant c.

	$y_1 = -2$	$y_1 = 1$	$y_1 = 4$
$y_2 = -2$	0.05	0.11	0.11
$y_2 = 1$	0.12	c	0.11
$y_2 = 4$	0.07	0.08	0.11

Give every probability to at least four significant figures.

(a) $c =$ _____

(b) The probability that the bank share beats the bond fund, $P(Y_1 > Y_2) =$ _____

(c) $P(Y_1 + Y_2 \le -1) =$ _____ (that is, an equal-weighted portfolio of the two returns at most -0.5 %)

Problem 8

seed 20260925

A bank has two small pools of SME loans. Over the next quarter let Y_1 be the number of defaults (0, 1 or 2) in pool A and Y_2 the number (0, 1 or 2) in pool B. The risk report does not give the joint probability function; it gives the joint distribution function $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$ at the grid points:

$F(y_1, y_2)$	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	0.32	0.38	0.48
$y_2 = 1$	0.36	0.52	0.72

$y_2 = 2$	0.47	0.70	1.00

Give every probability to at least four significant figures.

(a) $p(1,0) = P(Y_1 = 1, Y_2 = 0) =$ _____

(b) $p(1,1) = P(Y_1 = 1, Y_2 = 1) =$ _____

(c) $P(0 < Y_1 \le 2,\ 0 < Y_2 \le 1) =$ _____

Problem 9

seed 20260925

A Central Bank examiner audits a batch of 28 loan files at a commercial bank: 14 are performing, 10 are on the watch list and 4 are non-performing. She draws one file at random, sets it aside, and then draws a second file at random from the remaining 27.

Let Y_1 be the number of non-performing files and Y_2 the number of watch-list files among the two drawn.

Give every probability to at least four significant figures.

(a) $p(0,0) = P(Y_1 = 0, Y_2 = 0) =$ _____

(b) $p(1,1) =$ _____

(c) $p(2,0) =$ _____

(d) $F(1,0) = P(Y_1 \le 1, Y_2 \le 0) =$ _____

Problem 10

seed 20260925

A Baku bank's FX desk serves two large importers. On a given day let Y_1 and Y_2 be the numbers of large dollar-purchase orders (0, 1 or 2; the desk caps each client at two) placed by the first and second importer. The desk models their joint probability function as

$$p(y_1, y_2) = k\,(2\,y_1 + 10\,y_2 + 4), \qquad y_1 = 0,1,2, \quad y_2 = 0,1,2,$$

and $p(y_1, y_2) = 0$ otherwise.

Give every answer to at least four significant figures (a fraction is accepted).

(a) $k =$ _____

(b) $P(Y_1 > Y_2) =$ _____

(c) $F(1,1) =$ _____

(d) $P(Y_1 + Y_2 \ge 3) =$ _____

Problem 11

seed 20260925

A filling station on the Baku-Guba highway records its daily revenue Y_1 and daily profit Y_2, both in thousand AZN. Revenue never exceeds 5, and the profit margin never exceeds 28%, so $0 \le Y_2 \le 0.28\,Y_1$. The owner models the pair as uniform on that triangle:

$$f(y_1, y_2) = \begin{cases} k, & 0 \le y_1 \le 5,\ 0 \le y_2 \le 0.28\,y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ _____

(b) $P(Y_1 \le 2.5) =$ _____

(c) The probability that the day's margin is at most 22.4%, $P(Y_2 \le 0.224\,Y_1) =$ _____

(d) $P(Y_2 > 0.56) =$ _____

Problem 12

seed 20260925

A bank reloads one of its ATMs at the start of each week. Let Y_1 be the proportion of the machine's cash capacity that is loaded, and Y_2 the proportion of capacity withdrawn by customers during the week. Customers cannot withdraw more than was loaded, so $Y_2 \le Y_1$. As in Example 5.4, take

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $F(0.58, 0.40) = P(Y_1 \le 0.58,\ Y_2 \le 0.40) =$ _____

(b) The probability that at most 0.58 of capacity is loaded **and** more than 0.40 is withdrawn,
$P(Y_1 \le 0.58,\ Y_2 > 0.40) =$ _____

(c) $P(Y_2 > 0.4) =$ _____

Problem 13

seed 20260925

Two corporate clients each have an overdraft facility with the same bank. Let Y_1 and Y_2 be the proportions of their limits in use at the end of a month. The bank's model is

$$f(y_1, y_2) = \begin{cases} k\,(2.5\,y_1 + y_2), & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ _____

(b) $F(0.9, 0.4) =$ _____

(c) The probability that the first client is using a larger share of its limit than the second, $P(Y_1 > Y_2) =$ _____

(d) $P(Y_1 \le 0.9) =$ _____

Problem 14

seed 20260925

A distressed-debt fund holds two defaulted corporate bonds. Let Y_1 and Y_2 be the recovery rates (the fractions of face value eventually recovered) on the two bonds. The fund's model is

$$f(y_1, y_2) = \begin{cases} k\, y_1^2\, y_2^3, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give numerical answers to at least four significant figures.

(a) $k =$ _____

(b) $F(0.40, 0.80) = P(Y_1 \le 0.40,\ Y_2 \le 0.80) =$ _____

(c) For $0 \le t \le 1$, the probability that **both** recovery rates are at most t is $F(t, t) =$ _____ (enter a function of t; type powers with ^, for example t^2)

(d) The probability that both bonds recover more than these amounts, $P(Y_1 > 0.40,\ Y_2 > 0.80) =$ _____

Problem 15

seed 20260925

A bank's call centre has a retail desk and a business desk. Starting from 10:00, let Y_1 be the time until the next call reaches the retail desk and Y_2 the time until the next call reaches the business desk, both in minutes. The joint density is modelled as

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{16}\, e^{-(y_1 + y_2)/4}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(Y_1 < 10,\ Y_2 > 7) =$ _____

(b) The probability that the two waiting times add up to less than 18 minutes, $P(Y_1 + Y_2 < 18) =$ _____

Problem 16

seed 20260925

A bank models the times Y_1 and Y_2, in years from today, until two of its mortgage borrowers first miss a payment. Its model specifies the joint distribution function directly:

$$F(y_1, y_2) = \begin{cases} \left(1 - e^{-y_1/6}\right)\left(1 - e^{-y_2/15}\right), & y_1 > 0,\ y_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(2 < Y_1 \le 7,\ 2 < Y_2 \le 7) =$ _____

(b) The probability that the first borrower misses a payment within 2 years, whatever the second does, $P(Y_1 \le 2) =$ _____

(c) The probability that neither has missed a payment by the end of their windows in (a), $P(Y_1 > 7,\ Y_2 > 7) =$ _____

Problem 17

seed 20260925

A bank offers small firms an overdraft facility. For a randomly chosen firm and month, let Y_2 be the proportion of its overdraft limit that it draws, and Y_1 the proportion of the limit it repays before month end. A firm cannot repay more than it drew, so $Y_1 \le Y_2$. The bank's model is

$$f(y_1, y_2) = \begin{cases} k\,(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ _____

(b) $P(Y_1 \le 0.60,\ Y_2 \ge 0.50) =$ _____

(c) $P(Y_1 \le 0.20,\ Y_2 \ge 0.80) =$ _____

Problem 18

seed 20260925

A market maker watches two thinly traded stocks on the Baku Stock Exchange. From the opening bell let Y_1 be the time until the first block trade in stock A and Y_2 the time until the first block trade in stock B, in hours. The joint density is modelled as

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{33}\, e^{-y_1/3 - y_2/11}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) The probability that stock B trades first, $P(Y_1 > Y_2) =$ _____

(b) $P(Y_1 \le 3\,Y_2) =$ _____

(c) $P(Y_1 + Y_2 < 18) =$ _____

Problem 19

seed 20260925

A Baku engineering supplier bids in three public tenders that are decided one after another (tender 1, then 2, then 3) by three unrelated state buyers. It wins each tender with probability 0.40, independently of the others. The sales team's bonus depends on **when** the first win comes:

- 500 AZN if the first win is tender 1,
- 400 AZN if the first win is tender 2,
- 300 AZN if the first win is tender 3,
- a penalty of 400 AZN (bonus -400) if no tender is won.

Let Y_1 be the number of tenders won and Y_2 the bonus in AZN. Give every probability to at least four significant figures.

(a) $p(2,500) = P(Y_1 = 2, Y_2 = 500) =$ _____

(b) $p(1,400) =$ _____

(c) $F(2,400) = P(Y_1 \le 2, Y_2 \le 400) =$ _____

(d) $P(Y_1 \ge 2,\ Y_2 \ge 400) =$ _____

Problem 20

seed 20260925

A bank's FX desk runs under a gross risk limit of 2.5 million USD. At the close let Y_1 be the desk's net spot position in million USD (negative when the desk is short dollars) and $Y_2 \ge 0$ its options exposure in million USD. The limit requires $|Y_1| + Y_2 \le 2.5$. The risk unit models the pair as uniform on the triangle this allows, with vertices $(-2.5, 0)$, $(2.5, 0)$ and $(0, 2.5)$:

$$f(y_1, y_2) = \begin{cases} k, & y_2 \ge 0,\ |y_1| + y_2 \le 2.5, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ _____

(b) $P(Y_1 \le 2.25,\ Y_2 \le 1.625) =$ _____

(c) $P(Y_1 - Y_2 \ge 2) =$ _____