Week 12, Problem Set 1 - worked solutions

Wackerly 5.1-5.2 - Bivariate and Multivariate Probability Distributions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku bank has lent to two construction companies. Over the coming year let

- $Y_1 = 1$ if the first company defaults and $Y_1 = 0$ if it does not,
- $Y_2 = 1$ if the second company defaults and $Y_2 = 0$ if it does not.

The bank's credit-risk unit gives the joint probability function $p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2)$ below, but the entry for $(0, 0)$ has been left as an unknown constant c.

	$y_1 = 0$	$y_1 = 1$
$y_2 = 0$	c	0.13
$y_2 = 1$	0.07	0.02

Give every probability to at least four significant figures.

(a) The value of c that makes $p(y_1, y_2)$ a valid joint probability function: $c =$ ______________

(b) The probability that at least one of the two companies defaults: ______________

(c) The probability that exactly one of the two companies defaults: ______________

(a) By Theorem 5.1 every $p(y_1, y_2) \geq 0$ and the probabilities over all pairs (y_1, y_2) sum to 1:

$$c + 0.13 + 0.07 + 0.02 = 1 \quad \Longrightarrow \quad c = 1 - 0.22 = 0.78.$$

This is nonnegative, so the table is a genuine joint probability function. The cell $c = p(0, 0)$ is the probability that **neither** company defaults.

(b) "At least one defaults" is the union of the disjoint numerical events $(Y_1 = 1, Y_2 = 0)$, $(Y_1 = 0, Y_2 = 1)$ and $(Y_1 = 1, Y_2 = 1)$, so

$$p(1, 0) + p(0, 1) + p(1, 1) = 0.13 + 0.07 + 0.02 = 0.22.$$

Equivalently, it is the complement of $(Y_1 = 0, Y_2 = 0)$: $1 - 0.78 = 0.22$.

(c) Exactly one default means $(1, 0)$ or $(0, 1)$: $p(1, 0) + p(0, 1) = 0.13 + 0.07 = 0.20$.

Note that $p(1, 1) = 0.02$, the probability of a **joint** default, is the number a bank's capital planning cares about most: two losses in the same year. The remaining pairs with $Y_1 = Y_2$ have total probability $p(0, 0) + p(1, 1) = 0.80$.

Problem 2

seed 20260925

An insurer writes motor policies through its Ganja and Sumgait branches. On a given day let Y_1 be the number of large claims (0, 1 or 2) reported at Ganja and Y_2 the number reported at Sumgait. The actuaries' joint probability function $p(y_1, y_2)$ is below; the entry for $(0, 0)$ is an unknown constant c.

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	c	0.08	0.05
$y_2 = 1$	0.06	0.09	0.04
$y_2 = 2$	0.05	0.02	0.03

Give every probability to at least four significant figures.

(a) $c =$ ______________

(b) $P(1 \le Y_1 \le 2,\ Y_2 \le 1) =$ ______________

(c) The probability that both branches report the same number of large claims, $P(Y_1 = Y_2) =$ ______________

(a) Theorem 5.1: the nine probabilities are nonnegative and sum to 1, so

$$c = 1 - (0.08 + 0.05 + 0.06 + 0.09 + 0.04 + 0.05 + 0.02 + 0.03) = 1 - 0.42 = 0.58.$$

(b) The event $(1 \le Y_1 \le 2, Y_2 \le 1)$ is the union of the four disjoint pairs $(1,0), (1,1), (2,0), (2,1)$ (a rectangle in the table, just as in the book's calculation of $P(2 \le Y_1 \le 3, 1 \le Y_2 \le 2)$ for the dice):

$$p(1,0) + p(1,1) + p(2,0) + p(2,1) = 0.08 + 0.09 + 0.05 + 0.04 = 0.26.$$

(c) $Y_1 = Y_2$ picks out the diagonal of the table:

$$p(0,0) + p(1,1) + p(2,2) = 0.58 + 0.09 + 0.03 = 0.70.$$

Problem 3

seed 20260925

A customer holds two credit cards issued by different Baku banks. Over a quarter let Y_1 be the number of late payments (0, 1 or 2) on card A and Y_2 the number (0 or 1) on card B. Their joint probability function is

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	0.57	0.19	0.08
$y_2 = 1$	0.07	0.06	0.03

Let $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$ be the joint distribution function (Definition 5.2). Give every value to at least four significant figures.

(a) $F(1, 0) =$ ______________

(b) $F(0, 1) =$ ______________

(c) $F(1.7, 4) =$ ______________

(d) $F(2, 0.5) =$ ______________

For discrete variables Definition 5.2 gives $F(y_1, y_2) = \sum_{t_1 \le y_1} \sum_{t_2 \le y_2} p(t_1, t_2)$: add every cell whose y_1-label is at most the first argument **and** whose y_2-label is at most the second.

(a) $F(1,0)$: $t_1 \in \{0,1\}$, $t_2 = 0$, so $F(1,0) = p(0,0) + p(1,0) = 0.57 + 0.19 = 0.76$. This is the probability of at most one late payment on card A and none on card B.

(b) $F(0,1)$: $t_1 = 0$, $t_2 \in \{0,1\}$, so $F(0,1) = p(0,0) + p(0,1) = 0.57 + 0.07 = 0.64$.

(c) No value of Y_1 lies strictly between 1 and 1.7, so $Y_1 \le 1.7$ is the same event as $Y_1 \le 1$; and $Y_2 \le 4$ is certain. Hence

$$F(1.7, 4) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = 0.57 + 0.07 + 0.19 + 0.06 = 0.89.$$

(d) $Y_1 \le 2$ is certain and $Y_2 \le 0.5$ means $Y_2 = 0$, so $F(2, 0.5) = p(0,0) + p(1,0) + p(2,0) = 0.57 + 0.19 + 0.08 = 0.84$.

As in Example 5.2, F is a step function: it only changes when an argument crosses a value the variable can take, it is 0 when either argument is below 0, and it is 1 once $y_1 \ge 2$ and $y_2 \ge 1$.

Problem 4

seed 20260925

A bank's treasury desk expects two incoming wire transfers after the market opens. Let Y_1 be the arrival time of the first transfer and Y_2 that of the second, both in minutes after opening. The desk models the pair with the joint density

$$f(y_1, y_2) = \begin{cases} k, & 0 \le y_1 \le 20,\ 0 \le y_2 \le 40, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures (a fraction such as $1/500$ is also accepted).

(a) The value of k that makes f a joint density: $k =$ ______________

(b) $F(8, 32) = P(Y_1 \le 8,\ Y_2 \le 32) =$ ______________

(c) $P(4 \le Y_1 \le 18,\ Y_2 > 32) =$ ______________

(a) Theorem 5.2 requires $\int\int f = 1$. The density is the constant k over a rectangle of area $20 \times 40 = 800$, so

$$\int_0^{40}\int_0^{20} k\, dy_1\, dy_2 = 800\, k = 1 \quad \Longrightarrow \quad k = \frac{1}{800} \approx 0.001250.$$

(b) By Definition 5.3,

$$F(8, 32) = \int_0^{32}\int_0^{8} \frac{1}{800}\, dy_1\, dy_2 = \frac{8 \times 32}{800} = 0.3200.$$

As in Example 5.3, with a constant density the probability is (height) $\times$ (area of the region).

(c) The region is the rectangle $4 \le y_1 \le 18$, $32 < y_2 \le 40$ (the density is zero above 40):

$$\int_{32}^{40}\int_4^{18} \frac{1}{800}\, dy_1\, dy_2 = \frac{(18-4)(40-32)}{800} = \frac{14 \times 8}{800} = 0.1400.$$

Problem 5

seed 20260925

A corporate client has two revolving credit lines with a bank. Let Y_1 and Y_2 be the proportions of the two limits in use at month end. The bank models them with the joint density

$$f(y_1, y_2) = \begin{cases} k\, y_1 y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ ______________

(b) $F(0.25, 0.90) = P(Y_1 \le 0.25,\ Y_2 \le 0.90) =$ ______________

(c) $P(Y_1 \le 0.25,\ Y_2 > 0.90) =$ ______________

(d) The probability that more than 0.4 of the first line is in use, $P(Y_1 > 0.4) =$ ______________

(a) Theorem 5.2 requires the density to integrate to 1:

$$\int_0^1\int_0^1 k\, y_1 y_2\, dy_1\, dy_2 = k\left(\int_0^1 y_1\, dy_1\right)\left(\int_0^1 y_2\, dy_2\right) = k \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{k}{4} = 1,$$

so $k = 4$. Since $k > 0$, $f \ge 0$ everywhere, as the theorem also requires.

(b) For $0 \le y_1, y_2 \le 1$, Definition 5.3 gives

$$F(y_1, y_2) = \int_0^{y_2}\int_0^{y_1} 4 t_1 t_2\, dt_1\, dt_2 = \left[t_1^2\right]_0^{y_1}\left[t_2^2\right]_0^{y_2} = y_1^2\, y_2^2,$$

so $F(0.25, 0.90) = 0.0625 \times 0.8100 = 0.050625$.

(c) Integrate over $0 \le y_1 \le 0.25$, $0.90 < y_2 \le 1$:

$$\int_{0.90}^{1}\int_{0}^{0.25} 4y_1y_2\,dy_1\,dy_2 = 0.25^2\,(1-0.90^2) = 0.0625 \times (1-0.8100) = 0.011875.$$

Check: (b) + (c) $= 0.25^2 = P(Y_1 \le 0.25)$, because the two regions split the strip $y_1 \le 0.25$.

(d) The region is $0.4 < y_1 \le 1, 0 \le y_2 \le 1$:

$$\int_{0}^{1}\int_{0.4}^{1} 4y_1y_2\,dy_1\,dy_2 = (1-0.4^2)\cdot 1 = 1 - 0.16 = 0.84.$$

Problem 6

seed 20260925

A microfinance lender approves small-business loans with a ceiling of 20 thousand AZN. For a randomly chosen loan let Y_1 be the amount disbursed and Y_2 the amount repaid by the end of the first year, both in thousand AZN. Nothing can be repaid that was not disbursed, so $Y_2 \le Y_1$. The lender models the pair as uniform on the triangle this allows:

$$f(y_1, y_2) = \begin{cases} k, & 0 \le y_2 \le y_1 \le 20, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ ______________

(b) The probability that at most 18 thousand AZN is disbursed, $P(Y_1 \le 18) =$ ______________

(c) The probability that more than 8 thousand AZN is repaid, $P(Y_2 > 8) =$ ______________

Sketch the triangle with vertices $(0, 0)$, $(20, 0)$ and $(20, 20)$: it lies **below** the line $y_2 = y_1$.

(a) Theorem 5.2: $\int_0^{20}\int_0^{y_1} k\,dy_2\,dy_1 = k\int_0^{20} y_1\,dy_1 = k \cdot \frac{20^2}{2} = 1$, so $k = 2/20^2 = 2/400 \approx 0.005000$. (The area of the triangle is 200, and k is its reciprocal.)

(b) For $y_1 \le 18$ the variable y_2 runs from 0 to y_1:

$$P(Y_1 \le 18) = \int_0^{18}\int_0^{y_1} k\,dy_2\,dy_1 = k\,\frac{18^2}{2} = \frac{162}{200} = 0.8100.$$

(c) Because $y_2 \le y_1$, a repayment above 8 needs a disbursement above 8 too. So y_1 runs from 8 to 20, and for each such y_1, y_2 runs from 8 to y_1:

$$P(Y_2 > 8) = \int_8^{20}\int_8^{y_1} k\,dy_2\,dy_1 = k\int_8^{20}(y_1 - 8)\,dy_1 = k\,\frac{(20-8)^2}{2} = \frac{72}{200} = 0.3600.$$

With a constant density each probability is the fraction of the triangle's area that the region covers; both regions here are smaller triangles.

Problem 7

seed 20260925

An analyst models next month's returns on two assets traded in Baku: Y_1, the return on a bank share, and Y_2, the return on a government bond fund, both in percent. On a coarse grid each return is -2, 1 or 4, and the joint probability function is below. The entry for $(1, 1)$ is an unknown constant c.

	$y_1 = -2$	$y_1 = 1$	$y_1 = 4$
$y_2 = -2$	0.05	0.11	0.11
$y_2 = 1$	0.12	c	0.11
$y_2 = 4$	0.07	0.08	0.11

Give every probability to at least four significant figures.

(a) $c =$ ______________

(b) The probability that the bank share beats the bond fund, $P(Y_1 > Y_2) =$ ______________

(c) $P(Y_1 + Y_2 \le -1) =$ ______________ (that is, an equal-weighted portfolio of the two returns at most -0.5%)

(a) Theorem 5.1: the nine cells sum to 1, so $c = 1 - 0.76 = 0.24$.

(b) List the pairs (y_1, y_2) with $y_1 > y_2$: $(1, -2)$, $(4, -2)$ and $(4, 1)$. The event $(Y_1 > Y_2)$ is the union of these three disjoint numerical events, so

$$P(Y_1 > Y_2) = p(1, -2) + p(4, -2) + p(4, 1) = 0.11 + 0.11 + 0.11 = 0.33.$$

(c) Compute $y_1 + y_2$ for each cell:

	$y_1 = -2$	$y_1 = 1$	$y_1 = 4$
$y_2 = -2$	-4	-1	2
$y_2 = 1$	-1	2	5
$y_2 = 4$	2	5	8

The pairs with sum at most -1 are (-2,-2), (-2,1), (1,-2), so

$$P(Y_1 + Y_2 \le -1) = 0.05 + 0.12 + 0.11 = 0.28.$$

The equal-weighted portfolio returns $(Y_1 + Y_2)/2$, so the event is that the portfolio returns at most -0.5%. The joint table answers questions no single-asset distribution can: the portfolio's return depends on **which pairs** occur together.

Problem 8

seed 20260925

A bank has two small pools of SME loans. Over the next quarter let Y_1 be the number of defaults (0, 1 or 2) in pool A and Y_2 the number (0, 1 or 2) in pool B. The risk report does not give the joint probability function; it gives the joint distribution function $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$ at the grid points:

$F(y_1, y_2)$	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	0.32	0.38	0.48
$y_2 = 1$	0.36	0.52	0.72
$y_2 = 2$	0.47	0.70	1.00

Give every probability to at least four significant figures.

(a) $p(1, 0) = P(Y_1 = 1, Y_2 = 0) =$ ______________

(b) $p(1, 1) = P(Y_1 = 1, Y_2 = 1) =$ ______________

(c) $P(0 < Y_1 \le 2,\ 0 < Y_2 \le 1) =$ ______________

The tool is part 3 of the first Theorem 5.2: for $y_1^* \ge y_1$ and $y_2^* \ge y_2$,

$$P(y_1 < Y_1 \le y_1^*,\ y_2 < Y_2 \le y_2^*) = F(y_1^*, y_2^*) - F(y_1^*, y_2) - F(y_1, y_2^*) + F(y_1, y_2).$$

Subtracting the two "strips" removes the region below y_2 and the region left of y_1; their overlap was removed twice, so it is added back once.

(a) With $Y_2 = 0$ fixed there is only one strip to remove:
$p(1, 0) = P(Y_1 \le 1, Y_2 \le 0) - P(Y_1 \le 0, Y_2 \le 0) = F(1, 0) - F(0, 0) = 0.38 - 0.32 = 0.06$.

(b) The single cell $(1, 1)$ is the rectangle $0 < Y_1 \le 1$, $0 < Y_2 \le 1$:

$$p(1, 1) = F(1, 1) - F(1, 0) - F(0, 1) + F(0, 0) = 0.52 - 0.38 - 0.36 + 0.32 = 0.10.$$

(c) The rectangle $0 < Y_1 \le 2, 0 < Y_2 \le 1$ contains the cells $(1, 1)$ and $(2, 1)$:

$$F(2,1) - F(2,0) - F(0,1) + F(0,0) = 0.72 - 0.48 - 0.36 + 0.32 = 0.20.$$

As a check, $p(2, 1) = 0.10$ obtained the same way, and $0.10 + 0.10 = 0.20$. Every such difference must be nonnegative; a table of "cumulative" numbers that gave a negative rectangle could not be a joint distribution function.

Problem 9

seed 20260925

A Central Bank examiner audits a batch of 28 loan files at a commercial bank: 14 are performing, 10 are on the watch list and 4 are non-performing. She draws one file at random, sets it aside, and then draws a second file at random from the remaining 27.

Let Y_1 be the number of non-performing files and Y_2 the number of watch-list files among the two drawn.

Give every probability to at least four significant figures.

(a) $p(0, 0) = P(Y_1 = 0, Y_2 = 0) =$ ____________

(b) $p(1, 1) =$ ____________

(c) $p(2, 0) =$ ____________

(d) $F(1, 0) = P(Y_1 \le 1, Y_2 \le 0) =$ ____________

Follow Example 5.1: describe the sample points, give each its probability, and collect the sample points that make up each numerical event $(Y_1 = y_1, Y_2 = y_2)$.

A sample point is an ordered pair (first file, second file). By the mn rule there are $28 \times 27 = 756$ of them, and drawing at random makes them equally likely, each with probability $1/756$. Write P, W, D for performing, watch-list and non-performing.

(a) $(Y_1 = 0, Y_2 = 0)$ means both files are performing: the sample points (P, P). There are $14 \times 13 = 182$ of them, so $p(0, 0) = 182/756 = 0.2407$.

(b) $(Y_1 = 1, Y_2 = 1)$ is one non-performing and one watch-list file, in **either order**: (D, W) or (W, D). That is $4 \times 10 + 10 \times 4 = 80$ sample points, so $p(1, 1) = 80/756 = 0.1058$.

(c) $(Y_1 = 2, Y_2 = 0)$ is (D, D): $4 \times 3 = 12$ sample points, so $p(2, 0) = 12/756 = 0.0159$.

(d) By Definition 5.2, $F(1, 0) = p(0, 0) + p(1, 0)$. The event $(Y_1 = 1, Y_2 = 0)$ is one non-performing and one performing file, in either order: $2 \times 4 \times 14 = 112$ sample points. Hence

$$F(1,0) = \frac{182 + 112}{756} = \frac{294}{756} = 0.3889.$$

The same counts come out of combinations, e.g. $p(1, 1) = \binom{4}{1}\binom{10}{1}/\binom{28}{2}$; the ordered count simply doubles numerator and denominator.

Problem 10

seed 20260925

A Baku bank's FX desk serves two large importers. On a given day let Y_1 and Y_2 be the numbers of large dollar-purchase orders (0, 1 or 2; the desk caps each client at two) placed by the first and second importer. The desk models their joint probability function as

$$p(y_1, y_2) = k\,(2\,y_1 + 10\,y_2 + 4), \qquad y_1 = 0, 1, 2, \quad y_2 = 0, 1, 2,$$

and $p(y_1, y_2) = 0$ otherwise.

Give every answer to at least four significant figures (a fraction is accepted).

(a) $k =$ ____________

(b) $P(Y_1 > Y_2) =$ ____________

(c) $F(1,1) =$ ______________

(d) $P(Y_1 + Y_2 \geq 3) =$ ______________

Tabulate the weights $w(y_1, y_2) = 2y_1 + 10y_2 + 4$, so that $p(y_1, y_2) = k\, w(y_1, y_2)$:

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$	4	6	8
$y_2 = 1$	14	16	18
$y_2 = 2$	24	26	28

(a) All weights are positive, so part 1 of Theorem 5.1 holds for any $k > 0$. Part 2 fixes k: summing over the grid, $\sum y_1 = 3(0+1+2) = 9$ and likewise $\sum y_2 = 9$, while 4 is added nine times, so

$$\sum_{y_1, y_2} w(y_1, y_2) = 9(2) + 9(10) + 9(4) = 9(16) = 144, \qquad k = \frac{1}{144} \approx 0.006944.$$

(b) $y_1 > y_2$ at $(1,0), (2,0), (2,1)$: $P(Y_1 > Y_2) = (6 + 8 + 18)/144 = 32/144 = 0.2222$.

(c) $F(1,1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = (4 + 14 + 6 + 16)/144 = 40/144 = 0.2778$.

(d) $y_1 + y_2 \geq 3$ at $(1,2), (2,1), (2,2)$: $P(Y_1 + Y_2 \geq 3) = (26 + 18 + 28)/144 = 72/144 = 0.5000$.

Problem 11

seed 20260925

A filling station on the Baku-Guba highway records its daily revenue Y_1 and daily profit Y_2, both in thousand AZN. Revenue never exceeds 5, and the profit margin never exceeds 28%, so $0 \leq Y_2 \leq 0.28\, Y_1$. The owner models the pair as uniform on that triangle:

$$f(y_1, y_2) = \begin{cases} k, & 0 \leq y_1 \leq 5,\ 0 \leq y_2 \leq 0.28\, y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ ______________

(b) $P(Y_1 \leq 2.5) =$ ______________

(c) The probability that the day's margin is at most 22.4%, $P(Y_2 \leq 0.224\, Y_1) =$ ______________

(d) $P(Y_2 > 0.56) =$ ______________

The support is the triangle with vertices $(0,0)$, $(5,0)$ and $(5, 1.4)$. Sketch it before setting limits.

(a) Its area is $\frac{1}{2} \times 5 \times 1.4 = 3.5$, so by Theorem 5.2 $k \times 3.5 = 1$ and $k = 0.285714$. As a double integral: $\int_0^5 \int_0^{0.28 y_1} k\, dy_2\, dy_1 = k\, 0.28\, \frac{5^2}{2}$.

(b) $P(Y_1 \leq 2.5) = \int_0^{2.5} \int_0^{0.28 y_1} k\, dy_2\, dy_1 = k\, 0.28\, \frac{2.5^2}{2} = \left(\frac{2.5}{5}\right)^2 = 0.2500$.

(c) The line $y_2 = 0.224\, y_1$ lies inside the triangle, so

$$P(Y_2 \leq 0.224 Y_1) = \int_0^5 \int_0^{0.224 y_1} k\, dy_2\, dy_1 = k\, 0.224\, \frac{5^2}{2} = \frac{0.224}{0.28} = 0.8000.$$

(d) For $y_2 > 0.56$ we need $0.28 y_1 > 0.56$, i.e. $y_1 > 0.56/0.28 = 2$. So

$$P(Y_2 > 0.56) = \int_2^5 \int_{0.56}^{0.28 y_1} k\, dy_2\, dy_1 = k\, \frac{0.28}{2}\, (5-2)^2 = \left(1 - \frac{2}{5}\right)^2 = 0.3600.$$

Getting the lower limit on y_1 right in (d) is the whole problem: the region where the density is positive, not the rectangle $y_2 > 0.56$, decides where y_1 starts.

Problem 12

seed 20260925

A bank reloads one of its ATMs at the start of each week. Let Y_1 be the proportion of the machine's cash capacity that is loaded, and Y_2 the proportion of capacity withdrawn by customers during the week. Customers cannot withdraw more than was loaded, so $Y_2 \le Y_1$. As in Example 5.4, take

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $F(0.58, 0.40) = P(Y_1 \le 0.58,\ Y_2 \le 0.40) =$ ______________

(b) The probability that at most 0.58 of capacity is loaded **and** more than 0.40 is withdrawn, $P(Y_1 \le 0.58,\ Y_2 > 0.40) =$

____

(c) $P(Y_2 > 0.4) =$ ______________

The density is positive only on the triangle below the diagonal $y_2 = y_1$. In every part, intersect the event with that triangle first.

(a) $y_2 \le 0.40$ and $y_2 \le y_1 \le 0.58$. Integrating y_1 first, for each $y_2 \in [0, 0.40]$ the variable y_1 runs from y_2 to 0.58:

$$F(0.58, 0.40) = \int_0^{0.40} \int_{y_2}^{0.58} 3y_1 \, dy_1 \, dy_2 = \int_0^{0.40} \tfrac{3}{2}\left(0.58^2 - y_2^2\right) dy_2 = \tfrac{3}{2}\,(0.58)^2(0.40) - \tfrac{1}{2}\,(0.40)^3 = 0.169840.$$

(b) Now $0.40 < y_2 \le y_1 \le 0.58$, the small triangle of Example 5.4. For $y_1 \in (0.40, 0.58]$, y_2 runs from 0.40 to y_1:

$$\int_{0.40}^{0.58} \int_{0.40}^{y_1} 3y_1 \, dy_2 \, dy_1 = \int_{0.40}^{0.58} 3y_1(y_1 - 0.40) \, dy_1 = \left[y_1^3 - \tfrac{3}{2}(0.40)\, y_1^2\right]_{0.40}^{0.58} = 0.025272.$$

Check: (a) + (b) $= P(Y_1 \le 0.58) = \int_0^{0.58} 3y_1 \cdot y_1 \, dy_1 = 0.58^3 = 0.195112$.

(c) $y_2 > 0.4$ forces $y_1 > 0.4$:

$$P(Y_2 > 0.4) = \int_{0.4}^{1} \int_{0.4}^{y_1} 3y_1 \, dy_2 \, dy_1 = \left[y_1^3 - \tfrac{3}{2}(0.4)\, y_1^2\right]_{0.4}^{1} = 1 - \tfrac{3}{2}(0.4) + \tfrac{1}{2}(0.4)^3 = 0.4320.$$

Problem 13

seed 20260925

Two corporate clients each have an overdraft facility with the same bank. Let Y_1 and Y_2 be the proportions of their limits in use at the end of a month. The bank's model is

$$f(y_1, y_2) = \begin{cases} k\,(2.5\, y_1 + y_2), & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ ______________

(b) $F(0.9, 0.4) =$ ______________

(c) The probability that the first client is using a larger share of its limit than the second, $P(Y_1 > Y_2) =$ ______________

(d) $P(Y_1 \le 0.9) =$ ______________

(a) Theorem 5.2: $\int_0^1 \int_0^1 k(2.5y_1 + y_2) \, dy_1 \, dy_2 = k\left(2.5 \cdot \tfrac{1}{2} + \tfrac{1}{2}\right) = k \cdot 1.75 = 1$, so $k = 1/1.75 = 0.571429$. The integrand is nonnegative on the square.

(b) By Definition 5.3, for $0 \le y_1, y_2 \le 1$,

$$F(y_1,y_2)=\int_0^{y_2}\int_0^{y_1} k(2.5t_1+t_2)\,dt_1\,dt_2 = k\left(\frac{2.5\,y_1^2\,y_2}{2}+\frac{y_1\,y_2^2}{2}\right),$$

so

$$F(0.9,0.4)=0.571429\cdot\frac{2.5(0.9)^2(0.4)+(0.9)(0.4)^2}{2}=0.272571.$$

(c) $y_1>y_2$ is the triangle below the diagonal: for each $y_1\in[0,1]$, y_2 runs from 0 to y_1:

$$P(Y_1>Y_2)=\int_0^1\int_0^{y_1} k(2.5y_1+y_2)\,dy_2\,dy_1 = k\int_0^1 \left(2.5y_1^2+\tfrac{1}{2}y_1^2\right)dy_1 = k\left(\frac{2.5}{3}+\frac{1}{6}\right)=\frac{6}{10.5}=0.571429.$$

The coefficient $2.5>1$ tilts the density toward large y_1, so the answer exceeds $1/2$; with a coefficient of 1 the density would be symmetric and the probability exactly $1/2$.

(d) $P(Y_1\le 0.9)=F(0.9,1)=k\left(\frac{2.5(0.9)^2}{2}+\frac{0.9}{2}\right)=0.835714.$

Problem 14

seed 20260925

A distressed-debt fund holds two defaulted corporate bonds. Let Y_1 and Y_2 be the recovery rates (the fractions of face value eventually recovered) on the two bonds. The fund's model is

$$f(y_1,y_2)=\begin{cases} k\,y_1^2\,y_2^3, & 0\le y_1\le 1,\ 0\le y_2\le 1,\\ 0, & \text{elsewhere.}\end{cases}$$

Give numerical answers to at least four significant figures.

(a) $k=$ ______________

(b) $F(0.40,0.80)=P(Y_1\le 0.40,\ Y_2\le 0.80)=$ ______________

(c) For $0\le t\le 1$, the probability that **both** recovery rates are at most t is $F(t,t)=$ ______________ (enter a function of t; type powers with ^, for example t^2)

(d) The probability that both bonds recover more than these amounts, $P(Y_1>0.40,\ Y_2>0.80)=$ ______________

(a) The double integral over the square factors into two single integrals:

$$\int_0^1\int_0^1 k\,y_1^2y_2^3\,dy_1\,dy_2 = k\cdot\frac{1}{3}\cdot\frac{1}{4}=1 \quad\Longrightarrow\quad k=3\times 4=12.$$

(b) Definition 5.3: for $0\le y_1,y_2\le 1$,

$$F(y_1,y_2)=\int_0^{y_2}\int_0^{y_1} 12\,t_1^2t_2^3\,dt_1\,dt_2 = y_1^3\,y_2^4,$$

so $F(0.40,0.80)=0.40^3\times 0.80^4=0.064000\times 0.409600=0.026214.$

(c) Put $y_1=y_2=t$: $F(t,t)=t^3\,t^4=t^7$.

(d) The region $0.40<y_1\le 1$, $0.80<y_2\le 1$ is a rectangle, so

$$\int_{0.80}^1\int_{0.40}^1 12\,y_1^2y_2^3\,dy_1\,dy_2=(1-0.40^3)(1-0.80^4)=0.552614.$$

The same number comes from part 3 of Theorem 5.2 with $y_1^*=y_2^*=1$:
$F(1,1)-F(1,0.80)-F(0.40,1)+F(0.40,0.80)=1-0.409600-0.064000+0.026214.$

Problem 15

seed 20260925

A bank's call centre has a retail desk and a business desk. Starting from 10:00, let Y_1 be the time until the next call reaches the retail desk and Y_2 the time until the next call reaches the business desk, both in minutes. The joint density is modelled as

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{16}\, e^{-(y_1+y_2)/4}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(Y_1 < 10,\ Y_2 > 7) =$ ____________

(b) The probability that the two waiting times add up to less than 18 minutes, $P(Y_1 + Y_2 < 18) =$ ____________

Write $\theta = 4$, so $f(y_1, y_2) = \theta^{-2} e^{-y_1/\theta} e^{-y_2/\theta}$ on the positive quadrant. $\theta^2 = 16$, and the integral of f over the quadrant is 1, as Theorem 5.2 requires.

(a) The region $0 < y_1 < 10$, $y_2 > 7$ is a (semi-infinite) rectangle, so the double integral splits:

$$\int_7^\infty \int_0^{10} \frac{1}{\theta^2} e^{-y_1/\theta} e^{-y_2/\theta}\, dy_1\, dy_2 = \left(1 - e^{-10/\theta}\right) e^{-7/\theta} = (1 - 0.082085)(0.173774) = 0.159510.$$

(b) The region $y_1 + y_2 < 18$ in the quadrant is the triangle under the line $y_2 = 18 - y_1$. For $0 < y_1 < 18$, y_2 runs from 0 to $18 - y_1$:

$$\int_0^{18} \int_0^{18-y_1} \frac{1}{\theta^2} e^{-(y_1+y_2)/\theta}\, dy_2\, dy_1 = \int_0^{18} \frac{1}{\theta} e^{-y_1/\theta} \left(1 - e^{-(18-y_1)/\theta}\right) dy_1 = \left(1 - e^{-18/\theta}\right) - \frac{18}{\theta} e^{-18/\theta}.$$

In the last step the second term is $\int_0^{18} \theta^{-1} e^{-18/\theta}\, dy_1$, whose integrand does not depend on y_1. So

$$P(Y_1 + Y_2 < 18) = 1 - e^{-18/\theta} \left(1 + \frac{18}{\theta}\right) = 1 - 0.011109\,(1 + 4.500000) = 0.938901.$$

Problem 16

seed 20260925

A bank models the times Y_1 and Y_2, in years from today, until two of its mortgage borrowers first miss a payment. Its model specifies the joint distribution function directly:

$$F(y_1, y_2) = \begin{cases} \left(1 - e^{-y_1/6}\right)\left(1 - e^{-y_2/15}\right), & y_1 > 0,\ y_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Give every probability to at least four significant figures.

(a) $P(2 < Y_1 \le 7,\ 2 < Y_2 \le 7) =$ ____________

(b) The probability that the first borrower misses a payment within 2 years, whatever the second does, $P(Y_1 \le 2) =$ ____________

(c) The probability that neither has missed a payment by the end of their windows in (a), $P(Y_1 > 7,\ Y_2 > 7) =$ ____________

(a) Part 3 of Theorem 5.2 (the one on distribution functions) with $y_1 = 2, y_1^* = 7, y_2 = 2, y_2^* = 7$:

$$P(2 < Y_1 \le 7,\ 2 < Y_2 \le 7) = F(7,7) - F(7,2) - F(2,7) + F(2,2).$$

With $e^{-2/6} = 0.716531$, $e^{-7/6} = 0.311403$, $e^{-2/15} = 0.875173$, $e^{-7/15} = 0.627089$:

$$0.256785 - 0.085955 - 0.105709 + 0.035384 = 0.100506.$$

Because this F factors, the four terms collapse to $(0.716531 - 0.311403)(0.875173 - 0.627089)$, which is the same number.

(b) $(Y_1 \le 2)$ is the event $(Y_1 \le 2, Y_2 < \infty)$, so its probability is the limit of $F(2, y_2)$ as $y_2 \to \infty$:
$1 - e^{-2/6} = 1 - 0.716531 = 0.283469$.

(c) Use complements. $P(Y_1 > 7, Y_2 > 7) = 1 - P(Y_1 \le 7) - P(Y_2 \le 7) + F(7,7)$ (inclusion-exclusion on the events $Y_1 \le 7$ and $Y_2 \le 7$), which is part 3 of the theorem again with $y_1^* = y_2^* = \infty$. Substituting,

$$1 - (1 - 0.311403) - (1 - 0.627089) + (1 - 0.311403)(1 - 0.627089) = 0.311403 \times 0.627089 = 0.195278.$$

Problem 17

seed 20260925

A bank offers small firms an overdraft facility. For a randomly chosen firm and month, let Y_2 be the proportion of its overdraft limit that it draws, and Y_1 the proportion of the limit it repays before month end. A firm cannot repay more than it drew, so $Y_1 \le Y_2$. The bank's model is

$$f(y_1, y_2) = \begin{cases} k\,(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ ____________

(b) $P(Y_1 \le 0.60,\ Y_2 \ge 0.50) =$ ____________

(c) $P(Y_1 \le 0.20,\ Y_2 \ge 0.80) =$ ____________

The support is the triangle **above** the diagonal: vertices $(0,0)$, $(0,1)$, $(1,1)$ in the (y_1, y_2) plane. Integrate y_1 first: for a fixed y_2, y_1 runs from 0 to y_2.

(a) $\int_0^1 \int_0^{y_2} k(1 - y_2)\,dy_1\,dy_2 = k \int_0^1 (y_2 - y_2^2)\,dy_2 = k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6} = 1$, so $k = 6$.

(b) We need $y_1 \le 0.60$, $y_2 \ge 0.50$ and $y_1 \le y_2$. For a fixed y_2, y_1 runs from 0 to $\min(y_2, 0.60)$, and the minimum switches at $y_2 = 0.60$, which lies inside $[0.50, 1]$. So split the y_2-integral there:

$$\int_{0.50}^{0.60} 6(1 - y_2)\,y_2\,dy_2 + \int_{0.60}^{1} 6(1 - y_2)\,0.60\,dy_2.$$

The first piece is $\left[3y_2^2 - 2y_2^3\right]_{0.50}^{0.60} = 0.648000 - 0.500000 = 0.148000$. The second is $6(0.60) \cdot \frac{(1-0.60)^2}{2} = 3(0.60)(1 - 0.60)^2 = 0.288000$. Total: 0.436000.

(c) Now $0.20 < 0.80$, so for every $y_2 \ge 0.80$ we have $y_2 > 0.20$ and the minimum is always 0.20: y_1 runs over the full interval $[0, 0.20]$ and there is nothing to split.

$$\int_{0.80}^{1} 6(1 - y_2)\,0.20\,dy_2 = 3(0.20)(1 - 0.80)^2 = 0.024000.$$

The lesson is the one Example 5.4 makes with a sketch: the limits come from intersecting the event with the support, and the shape of that intersection depends on where the numbers fall.

Problem 18

seed 20260925

A market maker watches two thinly traded stocks on the Baku Stock Exchange. From the opening bell let Y_1 be the time until the first block trade in stock A and Y_2 the time until the first block trade in stock B, in hours. The joint density is modelled as

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{33}\,e^{-y_1/3 - y_2/11}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) The probability that stock B trades first, $P(Y_1 > Y_2) =$ ____________

(b) $P(Y_1 \le 3\,Y_2) =$ ____________

(c) $P(Y_1 + Y_2 < 18) =$ ______________

Write $\theta_1 = 3$, $\theta_2 = 11$, so $f(y_1, y_2) = \frac{1}{\theta_1} e^{-y_1/\theta_1} \cdot \frac{1}{\theta_2} e^{-y_2/\theta_2}$ and $\theta_1\theta_2 = 33$. In (a) and (b) integrate y_1 first, for fixed y_2.

(a) The region $y_1 > y_2 > 0$ is unbounded. For fixed y_2, y_1 runs from y_2 to ∞:

$$P(Y_1 > Y_2) = \int_0^\infty \frac{1}{\theta_2} e^{-y_2/\theta_2} \left(\int_{y_2}^\infty \frac{1}{\theta_1} e^{-y_1/\theta_1} dy_1 \right) dy_2 = \int_0^\infty \frac{1}{\theta_2} e^{-y_2\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)} dy_2 = \frac{1/\theta_2}{1/\theta_1 + 1/\theta_2} = \frac{\theta_1}{\theta_1+\theta_2} = \frac{3}{14} = 0.214286.$$

Stock A trades faster on average ($\theta_1 < \theta_2$), so B beats it less than half the time.

(b) For fixed y_2, y_1 runs from 0 to $3y_2$:

$$P(Y_1 \le 3Y_2) = \int_0^\infty \frac{1}{\theta_2} e^{-y_2/\theta_2} \left(1 - e^{-3y_2/\theta_1}\right) dy_2 = 1 - \frac{1/\theta_2}{3/\theta_1 + 1/\theta_2} = \frac{3\theta_2}{\theta_1 + 3\theta_2} = \frac{33}{36} = 0.916667.$$

(c) The triangle $y_1 + y_2 < 18$: for $0 < y_1 < 18$, y_2 runs from 0 to $18 - y_1$:

$$\int_0^{18} \frac{1}{\theta_1} e^{-y_1/\theta_1} \left(1 - e^{-(18-y_1)/\theta_2}\right) dy_1 = \left(1 - e^{-18/\theta_1}\right) - \frac{e^{-18/\theta_2}}{\theta_1} \int_0^{18} e^{-y_1\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)} dy_1.$$

Since $\frac{1}{\theta_1} - \frac{1}{\theta_2} = \frac{\theta_2-\theta_1}{\theta_1\theta_2} \ne 0$, the last integral is $\frac{\theta_1\theta_2}{\theta_2-\theta_1}\left(1 - e^{-18\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)}\right)$, and simplifying,

$$P(Y_1 + Y_2 < 18) = 1 - \frac{\theta_2 e^{-18/\theta_2} - \theta_1 e^{-18/\theta_1}}{\theta_2 - \theta_1} = 1 - \frac{11(0.194687) - 3(0.002479)}{8} = 0.733235.$$

Problem 19

seed 20260925

A Baku engineering supplier bids in three public tenders that are decided one after another (tender 1, then 2, then 3) by three unrelated state buyers. It wins each tender with probability 0.40, independently of the others. The sales team's bonus depends on **when** the first win comes:

- 500 AZN if the first win is tender 1,
- 400 AZN if the first win is tender 2,
- 300 AZN if the first win is tender 3,
- a penalty of 400 AZN (bonus -400) if no tender is won.

Let Y_1 be the number of tenders won and Y_2 the bonus in AZN. Give every probability to at least four significant figures.

(a) $p(2, 500) = P(Y_1 = 2, Y_2 = 500) =$ ______________

(b) $p(1, 400) =$ ______________

(c) $F(2, 400) = P(Y_1 \le 2, Y_2 \le 400) =$ ______________

(d) $P(Y_1 \ge 2,\ Y_2 \ge 400) =$ ______________

The sample points are the eight win/lose sequences, e.g. WLW = win tender 1, lose 2, win 3. By independence a sequence with j wins has probability $0.40^j\, 0.60^{3-j}$. Each sequence fixes both Y_1 and Y_2:

sequence	y_1	y_2	probability
WWW	3	500	0.064000
WWL, WLW	2	500	0.192000 (total)
WLL	1	500	0.144000
LWW	2	400	0.096000
LWL	1	400	0.144000
LLW	1	300	0.144000
LLL	0	-400	0.216000

These seven cells are the whole joint probability function; they sum to 1.

(a) $(Y_1 = 2, Y_2 = 500)$ contains WWL and WLW, so $p(2, 500) = 2(0.40)^2(0.60) = 0.192000$.

(b) $(Y_1 = 1, Y_2 = 400)$ is LWL alone: $(0.60)(0.40)(0.60) = 0.144000$.

(c) $Y_2 \le 400$ excludes every sequence starting with W. The remaining four, LWW, LWL, LLW and LLL, all have $Y_1 \le 2$, so

$$F(2, 400) = 0.096000 + 0.144000 + 0.144000 + 0.216000 = 0.600000.$$

Equivalently, $F(2, 400) = P(\text{tender 1 is lost}) = 0.60$.

(d) The cells with $y_1 \ge 2$ and $y_2 \ge 400$ are $(3, 500)$, $(2, 500)$ and $(2, 400)$:

$$0.064000 + 0.192000 + 0.096000 = 0.352000.$$

This is exactly $P(Y_1 \ge 2)$: two wins out of three force the first win to be tender 1 or tender 2, so the condition on Y_2 adds nothing. Seeing that from the table of sample points is the point of deriving the joint distribution.

Problem 20

seed 20260925

A bank's FX desk runs under a gross risk limit of 2.5 million USD. At the close let Y_1 be the desk's net spot position in million USD (negative when the desk is short dollars) and $Y_2 \ge 0$ its options exposure in million USD. The limit requires $|Y_1| + Y_2 \le 2.5$. The risk unit models the pair as uniform on the triangle this allows, with vertices $(-2.5, 0)$, $(2.5, 0)$ and $(0, 2.5)$:

$$f(y_1, y_2) = \begin{cases} k, & y_2 \ge 0,\ |y_1| + y_2 \le 2.5, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures.

(a) $k =$ ______________

(b) $P(Y_1 \le 2.25,\ Y_2 \le 1.625) =$ ______________

(c) $P(Y_1 - Y_2 \ge 2) =$ ______________

With a constant density every probability is (area of the region inside the triangle) $\times\ k$, so the work is geometry once the region is sketched.

(a) The triangle has base 2×2.5 and height 2.5, so area $2.5^2 = 6.25$. Theorem 5.2 gives $k \times 6.25 = 1$, $k = 0.160000$.

(b) Start from the whole triangle and remove what violates each condition.

- $Y_2 > 1.625$: the small triangle at the top, with vertices $(\pm 0.875, 1.625)$ and $(0, 2.5)$; base $2(0.875)$, height 0.875, area $(0.875)^2 = 0.765625$.
- $Y_1 > 2.25$: the triangle at the right with vertices $(2.25, 0)$, $(2.5, 0)$, $(2.25, 0.25)$; area $\frac{1}{2}(0.25)^2 = 0.031250$. Since $0.25 < 1.625$, this piece lies entirely **below** $y_2 = 1.625$, so it does not overlap the first piece.

$$P(Y_1 \le 2.25, Y_2 \le 1.625) = \frac{6.25 - 0.765625 - 0.031250}{6.25} = \frac{5.453125}{6.25} = 0.872500.$$

(Had $2.5 - 2.25$ exceeded 1.625, the two pieces would overlap and the right-hand piece would have to be cut at $y_2 = 1.625$.)

(c) $y_1 - y_2 \ge 2$ is the region on or below the line $y_2 = y_1 - 2$. Inside the support it is the triangle bounded by $y_2 = 0$, that line, and the edge $y_2 = 2.5 - y_1$. Its vertices are $(2, 0)$, $(2.5, 0)$ and the intersection of the two lines, $(2.25, 0.25)$. Base 0.5, height 0.25:

$$P(Y_1 - Y_2 \ge 2) = \frac{\frac{1}{2}(0.5)(0.25)}{6.25} = \frac{0.062500}{6.25} = 0.010000.$$

As a double integral: $\int_0^{0.25} \int_{y_2+2}^{2.5-y_2} k\, dy_1\, dy_2$, which gives the same number.

Answer key

Problem	Answers
1	(a) 0.78, (b) 0.22, (c) 0.2
2	(a) 0.58, (b) 0.26, (c) 0.7
3	(a) 0.76, (b) 0.64, (c) 0.89, (d) 0.84
4	(a) 0.00125, (b) 0.32, (c) 0.14
5	(a) 4, (b) 0.050625, (c) 0.011875, (d) 0.84
6	(a) 0.005, (b) 0.81, (c) 0.36
7	(a) 0.24, (b) 0.33, (c) 0.28
8	(a) 0.06, (b) 0.1, (c) 0.2
9	(a) 0.240741, (b) 0.10582, (c) 0.015873, (d) 0.388889
10	(a) 0.00694444, (b) 0.222222, (c) 0.277778, (d) 0.5
11	(a) 0.285714, (b) 0.25, (c) 0.8, (d) 0.36
12	(a) 0.16984, (b) 0.025272, (c) 0.432
13	(a) 0.571429, (b) 0.272571, (c) 0.571429, (d) 0.835714
14	(a) 12, (b) 0.0262144, (c) t^7, (d) 0.552614
15	(a) 0.15951, (b) 0.938901
16	(a) 0.100506, (b) 0.283469, (c) 0.195278
17	(a) 6, (b) 0.436, (c) 0.024
18	(a) 0.214286, (b) 0.916667, (c) 0.733235
19	(a) 0.192, (b) 0.144, (c) 0.6, (d) 0.352
20	(a) 0.16, (b) 0.8725, (c) 0.01