Week 12, Problem Set 2

Wackerly 5.3 - Marginal and Conditional Probability Distributions

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku household budget survey classifies each household by its income band Y_1 and its monthly consumption band Y_2, each coded 1 = low, 2 = middle, 3 = high. For a randomly chosen household the joint probability function $p(y_1, y_2)$ is

$y_2 \backslash y_1$	1	2	3
1	0.07	0.05	0.04
2	0.04	0.46	0.07
3	0.04	0.08	0.15

(rows are values of y_2, columns are values of y_1).

(a) $p_1(1) = P(Y_1 = 1) =$ ____________

(b) $p_1(3) = P(Y_1 = 3) =$ ____________

(c) $p_2(1) = P(Y_2 = 1) =$ ____________

(d) $p_2(3) = P(Y_2 = 3) =$ ____________

(e) $P(Y_1 \geq 2, Y_2 \geq 2) =$ ____________

Problem 2

seed 20260925

A bank's consumer-credit book is classified by loan size Y_1 (1 = small, 2 = medium, 3 = large) and by outcome Y_2 (1 if the loan defaulted within two years, 0 if it did not). For a loan drawn at random from the book, the joint probability function $p(y_1, y_2)$ is

$y_2 \backslash y_1$	1	2	3
0	0.38	0.36	0.17
1	0.02	0.05	0.02

Give every probability to at least four significant figures.

(a) The marginal default probability $p_2(1) = P(Y_2 = 1) =$ ____________

(b) The default rate among large loans, $P(Y_2 = 1 \mid Y_1 = 3) =$ ____________

(c) The share of defaulted loans that were large, $P(Y_1 = 3 \mid Y_2 = 1) =$ ____________

(d) $P(Y_2 = 0 \mid Y_1 = 1) =$ ______________

Problem 3

seed 20260925

A bank monitors two revolving credit lines of the same corporate client. Let Y_1 and Y_2 be the proportions of the two limits that are drawn at the end of a randomly chosen month. The bank's model for their joint density is

$$f(y_1, y_2) = \begin{cases} 0.60\, y_1 + 1.40\, y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) For $0 \le y_1 \le 1$, the marginal density of Y_1 is $f_1(y_1) =$ ______________

(a formula in y_1; type it in the form $3 * y_1 + 0.2$)

(b) $P(Y_1 \le 0.85) =$ ______________

(c) $P(Y_2 > 0.40) =$ ______________

Give probabilities to at least four significant figures.

Problem 4

seed 20260925

A mobile operator in Azerbaijan studies postpaid billing. For a randomly chosen postpaid account let Y_2 be the segment ($1 =$ consumer, $2 =$ business) and Y_1 the number of bills paid late in the last quarter. The joint probability function $p(y_1, y_2)$ is

$y_2 \backslash y_1$	0	1	2	3
1	0.28	0.15	0.05	0.04
2	0.35	0.05	0.05	0.03

Give every probability to at least four significant figures.

(a) The conditional probability function of Y_1 given $Y_2 = 2$:

$p(0 \mid 2) =$ ______________

$p(1 \mid 2) =$ ______________

$p(2 \mid 2) =$ ______________

(b) The probability that a consumer account paid at least one bill late, $P(Y_1 \ge 1 \mid Y_2 = 1) =$ ______________

Problem 5

seed 20260925

An underground gas storage facility near Baku has working capacity 70 million cubic metres. Let Y_2 be the volume in storage at the start of a winter month and Y_1 the volume withdrawn during the month (both in million cubic metres). The facility is not refilled during the month, so $Y_1 \le Y_2$. The operator models (Y_1, Y_2) as uniformly distributed over the triangle $0 \le y_1 \le y_2 \le 70$:

$$f(y_1, y_2) = \begin{cases} k, & 0 \le y_1 \le y_2 \le 70, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every answer to at least four significant figures (decimals, not fractions of large numbers, are fine).

(a) The constant $k =$ ______________

(b) The marginal density of Y_2 at $y_2 = 56$: $f_2(56) =$ ______________

(c) $P(Y_1 \le 13 \mid Y_2 = 56) =$ ______________

(d) $P(Y_1 \le 13 \mid Y_2 = 70) =$ ______________

Problem 6

seed 20260925

On the Baku Stock Exchange, let Y_1 and Y_2 be the times (in minutes, from 11:00) until the next trade in stock A and in stock B. A market-microstructure analyst uses the joint density

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{25.50}\, e^{-y_1/3.0 \,-\, y_2/8.5}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(Note $25.50 = 3.0 \times 8.5$.) Give probabilities to at least four significant figures.

(a) For $y_1 > 0$, $f_1(y_1) =$ ______________

(a formula in y_1; use $e^(\ldots)$ or $exp(\ldots)$)

(b) $P(2.0 < Y_1 < 6.5) =$ ______________

(c) $P(2.0 < Y_1 < 6.5 \mid Y_2 = 5) =$ ______________

(d) $P(Y_2 > 11) =$ ______________

Problem 7

seed 20260925

A consumer-lending unit uses a household survey in which Y_1 is the household's income band and Y_2 its consumption band, each coded $1 =$ low, $2 =$ middle, $3 =$ high. The joint probability function $p(y_1, y_2)$ is

$y_2 \backslash y_1$	1	2	3
1	0.09	0.07	0.03
2	0.05	0.41	0.08
3	0.03	0.10	0.14

Give every probability to at least four significant figures.

(a) The conditional probability function of consumption for high-income households:

$P(Y_2 = 1 \mid Y_1 = 3) =$ ______________

$P(Y_2 = 2 \mid Y_1 = 3) =$ ______________

$P(Y_2 = 3 \mid Y_1 = 3) =$ ______________

(b) Among high-consumption households, the probability of high income, $P(Y_1 = 3 \mid Y_2 = 3) =$

____

(c) $P(Y_2 = 3 \mid Y_1 = 1) =$ ______________

Problem 8

seed 20260925

A Baku commercial bank's loan book, by size band Y_1 ($1 =$ micro, $2 =$ SME, $3 =$ corporate) and status Y_2 ($0 =$ current, $1 =$ 30-89 days past due, $2 =$ defaulted), contains the following **numbers of loans**:

$y_2 \backslash y_1$	1	2	3
0	480	470	180
1	67	25	11
2	28	32	2

A supervisor draws one loan at random from the whole book, so that $p(y_1, y_2)$ is the count in cell (y_1, y_2) divided by the total. Give every probability to at least four significant figures.

(a) $p_1(3) = P(Y_1 = 3) =$ ______________

(b) The default rate of SME loans, $P(Y_2 = 2 \mid Y_1 = 2) =$ ______________

(c) The SME share of defaulted loans, $P(Y_1 = 2 \mid Y_2 = 2) =$ ______________

(d) $P(Y_2 \geq 1 \mid Y_1 = 1) =$ ______________

Problem 9

seed 20260925

A Central Bank examiner audits a branch whose small-business loan files consist of 6 performing loans, 8 watch-list loans and 4 non-performing loans (18 files in total). She selects 3 files at random, without replacement. Let Y_1 be the number of performing and Y_2 the number of watch-list files in her sample, so that

$$p(y_1, y_2) = \frac{\binom{6}{y_1}\binom{8}{y_2}\binom{4}{3-y_1-y_2}}{\binom{18}{3}}, \qquad y_1, y_2 \geq 0,\ y_1 + y_2 \leq 3.$$

Give every probability to at least four significant figures.

(a) The marginal probability $p_1(3) = P(Y_1 = 3) =$ ______________

(b) $P(Y_1 = 1 \mid Y_2 = 1) =$ ______________

(c) $P(Y_1 = 2 \mid Y_2 = 0) =$ ______________

(d) $p_2(0) = P(Y_2 = 0) =$ ______________

Problem 10

seed 20260925

A fuel depot restocks its storage tank on Monday mornings. Let Y_1 be the proportion of the tank's capacity stocked on Monday and Y_2 the proportion of capacity sold during the week, so $Y_2 \leq Y_1$. The joint density is

$$f(y_1, y_2) = \begin{cases} 3\,y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) For $0 \leq y_2 \leq 1$, the marginal density of Y_2 is $f_2(y_2) =$ ______________

(a formula in y_2)

(b) Given that the depot stocked 0.90 of capacity, the probability that it sells more than 0.25 of capacity:
$P(Y_2 > 0.25 \mid Y_1 = 0.90) =$ ______________

(c) Given that it sold 0.45 of capacity, the probability that it had stocked more than 0.70:
$P(Y_1 > 0.70 \mid Y_2 = 0.45) =$ ______________

Give probabilities to at least four significant figures.

Problem 11

seed 20260925

A ministry tracks capital projects over a fiscal year. For a randomly chosen project let Y_2 be the proportion of its budget **committed** (contracts signed) by the end of Q3 and Y_1 the proportion actually **disbursed**; nothing is disbursed before it is committed, so $Y_1 \le Y_2$. The joint density is

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) For $0 \le y_1 \le 1$, $f_1(y_1) =$ ______________ (a formula in y_1)

(b) $P(Y_2 \le 0.50 \mid Y_1 \le 0.40) =$ ______________

(c) $P(Y_2 \ge 0.60 \mid Y_1 = 0.35) =$ ______________

Give probabilities to at least four significant figures.

Problem 12

seed 20260925

A bank's FX department runs two market-making desks. Over a randomly chosen trading day let Y_1 and Y_2 be the proportions of the session during which desk I and desk II are actively quoting two-way prices. Their joint density is

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every probability to at least four significant figures.

(a) On a day when desk II quotes for exactly 0.85 of the session, the probability that desk I quotes for more than 0.30 of it: $P(Y_1 > 0.30 \mid Y_2 = 0.85) =$ ______________

(b) $P(Y_1 \ge 0.45 \mid Y_2 \ge 0.75) =$ ______________

(c) Unconditionally, $P(Y_1 > 0.30) =$ ______________

Problem 13

seed 20260925

For a consumer-loan application at a Baku bank let Y_1 be the total time (in working days) from submission to disbursement and Y_2 the time the file waits in the underwriting queue before anyone opens it, so $Y_2 \le Y_1$. The bank's operations team fits

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{9.00}\, e^{-y_1/3.0}, & 0 \le y_2 \le y_1 < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

where $9.00 = 3.0^2$. Give probabilities to at least four significant figures.

(a) For $y_1 > 0$, $f_1(y_1) =$ ______________ (a formula in y_1; use $e^(\dots)$ or $exp(\dots)$)

(b) $P(Y_2 > 6) =$ ______________

(c) An application took 8 days in total. The probability that it waited in the queue less than 3.5 days: $P(Y_2 < 3.5 \mid Y_1 = 8) =$ ______________

(d) An application waited 4 days in the queue. The probability that it is still not disbursed 6 days after leaving the queue: $P(Y_1 > 10 \mid Y_2 = 4) =$ ______________

Problem 14

seed 20260925

Each morning a bank's treasury allocates a trading desk a fraction Y_2 of the desk's maximum position limit; over many days Y_2 is uniformly distributed on $(0, 1)$. Given the allocation $Y_2 = y_2$, the fraction of the maximum limit the desk actually uses that day, Y_1, is uniformly distributed on $(0, y_2)$.

Give every answer to at least four significant figures.

(a) The joint density of (Y_1, Y_2) at the point $y_1 = 0.25$, $y_2 = 0.40$: $f(0.25, 0.40) =$ ______________

(b) $P(Y_1 > 0.32 \mid Y_2 = 0.60) =$ ______________

(c) The unconditional probability $P(Y_1 \le 0.35) =$ ______________

(d) On a day when the desk used 0.40 of the maximum limit, the probability that it had been allocated more than 0.85: $P(Y_2 > 0.85 \mid Y_1 = 0.40) =$ ______________

Problem 15

seed 20260925

A corporate treasury hedges its foreign-currency receivables. For a randomly chosen month let Y_1 be the gross open exposure and Y_2 the net exposure after hedging (both in million USD). Gross exposure never exceeds 19, and policy requires the hedge to cover enough that $2.0\, Y_2 \le Y_1$. The risk team models (Y_1, Y_2) as uniformly distributed over the region the policy allows:

$$f(y_1, y_2) = \begin{cases} \dfrac{2(2.0)}{19^2}, & 0 \le y_1 \le 19,\ 0 \le y_2,\ 2.0\, y_2 \le y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) For $0 \le y_2 \le 19/2.0$, the marginal density of net exposure $f_2(y_2) =$ ______________ (a formula in y_2)

(b) $P(Y_2 > 0.60) =$ ______________

(c) In a month when net exposure was 7.60, the probability that gross exposure exceeded 18.35: $P(Y_1 > 18.35 \mid Y_2 = 7.60) =$ ______________

Give probabilities to at least four significant figures.

Problem 16

seed 20260925

A regulator samples private pension funds at random. For a fund let Y_1 be the share of assets held in equities and Y_2 the share held in bonds; the rest is cash and property, so $Y_1 + Y_2 \le 1$. Across funds the joint density is

$$f(y_1, y_2) = \begin{cases} 6\,y_1, & y_1 \ge 0,\ y_2 \ge 0,\ y_1 + y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) For $0 \le y_2 \le 1$, $f_2(y_2) =$ ______________ (a formula in y_2)

(b) For a fund with exactly 0.15 in bonds, the probability of at least 0.40 in equities:
$P(Y_1 \ge 0.40 \mid Y_2 = 0.15) =$ ______________

(c) For a fund with **at most** 0.20 in bonds, the probability of at least 0.50 in equities:
$P(Y_1 \ge 0.50 \mid Y_2 \le 0.20) =$ ______________

Give probabilities to at least four significant figures.

Problem 17

seed 20260925

Electricity distributors in a region set an off-peak tariff Y_1 and a peak tariff Y_2 (both in AZN per 10 kWh). The peak tariff is never below the off-peak one, and the regulator caps the sum of the two at 2. Across distributors the tariff pair has joint density

$$f(y_1, y_2) = \begin{cases} 6\,y_1^2\,y_2, & 0 \le y_1 \le y_2,\ y_1 + y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Give probabilities to at least four significant figures.

(a) For $0 \le y_1 \le 1$, $f_1(y_1) =$ ______________ (a formula in y_1)

(b) $P(Y_2 \le 0.75) =$ ______________

(c) $P(Y_2 < 0.93 \mid Y_1 = 0.65) =$ ______________

(d) $P(Y_1 \le 0.23 \mid Y_2 = 1.45) =$ ______________

Problem 18

seed 20260925

A property insurer writes policies whose limits (maximum payouts) range from 15 to 100 thousand AZN. For a randomly chosen claim, the limit Y_2 of the policy it came from is uniformly distributed on $(15, 100)$, and given $Y_2 = y_2$ the claim amount Y_1 (thousand AZN) is uniformly distributed on $(0, y_2)$.

Give every answer to at least four significant figures.

(a) For a policy with limit 49.0, the probability that the claim exceeds 13.5: $P(Y_1 > 13.5 \mid Y_2 = 49.0) =$

____

(b) The marginal density of the claim amount at $y_1 = 7.5$: $f_1(7.5) =$ ______________

(c) $P(Y_1 > 83.0) =$ ______________

(d) A claim of 7.5 thousand AZN has arrived. The probability that the policy's limit exceeds 74.5:
$P(Y_2 > 74.5 \mid Y_1 = 7.5) =$ ______________

Problem 19

seed 20260925

A bank's anti-fraud system raises alerts at two branches. The weekly numbers of alerts, Y_1 at branch 1 and Y_2 at branch 2, are independent Poisson random variables with means $\lambda_1 = 2.0$ and $\lambda_2 = 6.0$. Let $W = Y_1 + Y_2$; you may use the fact (proved in Chapter 6) that W is Poisson with mean $\lambda_1 + \lambda_2$.

This week there were $W = 6$ alerts in total. Give every probability to at least four significant figures.

(a) Given $W = 6$, the conditional distribution of Y_1 is binomial with $n = 6$ and success probability $p =$

____

(b) $P(Y_1 = 3 \mid W = 6) =$ ______________

(c) The probability that branch 1 raised at least half of this week's alerts, $P(Y_1 \geq 3 \mid W = 6) =$

____

Problem 20

seed 20260925

A credit analyst is unsure of the one-year default probability Θ of small firms in the retail sector. She models it as a random variable with density

$$f(\theta) = 9(1-\theta)^8, \qquad 0 < \theta < 1,$$

(a beta density with $\alpha = 1$, $\beta = 9$, mean 0.1000). A bank holds loans to 28 such firms; given $\Theta = \theta$, the number Y of them that default is binomial with $n = 28$ and probability θ, so the joint distribution of (Y, Θ) is $p(y \mid \theta)\, f(\theta)$.

Give every probability to at least four significant figures.

(a) Before any data, $P(\Theta > 0.11) =$ ______________

(b) The marginal probability that none of the 28 firms defaults, $P(Y = 0) =$ ______________

(c) The marginal probability of exactly one default, $P(Y = 1) =$ ______________

(d) A year passes with no defaults. The conditional probability that the sector's default probability exceeds 0.11:
$P(\Theta > 0.11 \mid Y = 0) =$ ______________