# Week 13, Problem Set 1

Wackerly 5.4 - Independent Random Variables

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

## Problem 1

seed 20260925

A bank flags every cash withdrawal above 20,000 AZN for a compliance check. Let $Y_1$ be the number of flagged withdrawals at its Ganja branch tomorrow and $Y_2$ the number at its Lankaran branch. The two branches serve separate customer bases in cities 300 km apart, and the compliance team models $Y_1$ and $Y_2$ as **independent** random variables with marginal probability functions

| $y$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_1(y)$ | 0.43 | 0.29 | 0.28 |
| $p_2(y)$ | 0.38 | 0.33 | 0.29 |

(no branch ever flags more than two). Give every probability to at least four significant figures.

**(a)** $p(0,0) = P(Y_1 = 0, Y_2 = 0) =$ ______________

**(b)** $p(2,1) =$ ______________

**(c)** The probability that exactly two withdrawals are flagged in total, $P(Y_1 + Y_2 = 2) =$ ______________

**(d)** The probability that the two branches flag the same number, $P(Y_1 = Y_2) =$ ______________

## Problem 2

seed 20260925

A bank's card division records two indicators for each credit-card customer: $Y_1 = 1$ if the customer paid late last month ($0$ otherwise) and $Y_2 = 1$ if the customer holds a savings deposit with the bank ($0$ otherwise). For a randomly chosen customer the joint probability function is

| | $y_2 = 0$ | $y_2 = 1$ |
|---|---|---|
| $y_1 = 0$ | 0.3654 | 0.5046 |
| $y_1 = 1$ | 0.0546 | 0.0754 |

Give every probability to at least four significant figures.

**(a)** $p_1(1) = P(Y_1 = 1) =$ ______________

**(b)** $p_2(1) = P(Y_2 = 1) =$ ______________

**(c)** $p_1(1)\,p_2(1) =$ ______________

**(d)** Compare (c) with $p(1,1)$. Are $Y_1$ and $Y_2$ independent?

○ Independent: p(y1, y2) = p1(y1) p2(y2) in every cell

○ Dependent: the product rule fails in at least one cell

## Problem 3

seed 20260925

An insurer runs two unrelated claim desks: motor claims at its Ganja office and health claims at its Sheki office. Starting at 9:00 tomorrow, let $Y_1$ be the time, in hours, until the next motor claim arrives in Ganja and $Y_2$ the time until the next health claim arrives in Sheki. The insurer treats $Y_1$ and $Y_2$ as independent, with $Y_1$ exponential with mean $\beta_1 = 2.5$ and $Y_2$ exponential with mean $\beta_2 = 9.5$.

Give every probability to at least four significant figures.

**(a)** $P(Y_1 > 2.5) =$ ______________

**(b)** The probability that neither desk has a claim by its threshold, $P(Y_1 > 2.5,\ Y_2 > 7.5) =$ ______________

**(c)** $P(Y_1 \le 2.5,\ Y_2 \le 7.5) =$ ______________

## Problem 4

seed 20260925

A mobile operator monitors two base-station towers in different regions, on separate power grids. Let $Y_1$ be the fraction of tomorrow that tower A spends at peak load and $Y_2$ the time, in days, until tower B's next outage. Their marginal distribution functions are

$$F_1(y_1) = y_1^2, \quad 0 \le y_1 \le 1, \qquad\qquad F_2(y_2) = 1 - e^{-y_2/29}, \quad y_2 \ge 0,$$

(with $F_1 = 0$ below 0 and $1$ above 1, and $F_2 = 0$ for $y_2 < 0$), and the operator models $Y_1$ and $Y_2$ as independent in the sense of Definition 5.8.

Give every probability to at least four significant figures.

**(a)** The joint distribution function at the point $(0.70, 5)$: $F(0.70, 5) = P(Y_1 \le 0.70,\ Y_2 \le 5) =$

______________

**(b)** $P(0.25 < Y_1 \le 0.70,\ 5 < Y_2 \le 16) =$ ______________

## Problem 5

seed 20260925

For a small manufacturing firm, let $Y_1$ be the fraction of its bank credit line drawn at the end of a quarter and $Y_2$ the fraction of its customer invoices paid on time during that quarter. A lender's model gives the joint density

$$f(y_1, y_2) = \begin{cases} c\, y_1\, y_2^4, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** The constant $c =$ ______________

Theorem 5.5 shows $Y_1$ and $Y_2$ are independent, which you may use below. Give each probability to at least four significant figures.

**(b)** $P(Y_1 \le 0.45) =$ ______________

**(c)** $P(Y_1 \le 0.45,\ Y_2 > 0.75) =$ ______________

## Problem 6

seed 20260925

A bank classifies each loan application by segment, $Y_1 = 0$ (retail) or $Y_1 = 1$ (SME), and by channel, $Y_2 = 1$ (mobile app), $2$ (branch) or $3$ (sales agent). A fraction $P(Y_1 = 1) = 0.25$ of applications come from SMEs, and the conditional probability functions of the channel given the segment are

| | $y_2 = 1$ | $y_2 = 2$ | $y_2 = 3$ |
|---|---|---|---|
| $p(y_2 \mid Y_1 = 0)$ | 0.58 | 0.23 | 0.19 |
| $p(y_2 \mid Y_1 = 1)$ | 0.51 | 0.23 | 0.26 |

Give every probability to at least four significant figures.

**(a)** $p(1,1) = P(Y_1 = 1, Y_2 = 1) =$ ______________

**(b)** $p_2(1) = P(Y_2 = 1) =$ ______________

**(c)** Are $Y_1$ and $Y_2$ independent?

◯ Independent: the conditional distribution of Y2 is the same in both rows
◯ Dependent: the conditional distribution of Y2 changes with Y1

## Problem 7

seed 20260925

A credit bureau in Baku summarises its files on individual borrowers by $Y_1$, the number of late payments in the last year (0, 1, or 2 or more, coded 2), and $Y_2$, the number of credit products held (1, 2, or 3). For a randomly chosen file the joint probability function is

| | $y_2 = 1$ | $y_2 = 2$ | $y_2 = 3$ |
|---|---|---|---|
| $y_1 = 0$ | 0.1750 | 0.1500 | 0.1750 |
| $y_1 = 1$ | 0.0875 | 0.0750 | 0.0875 |
| $y_1 = 2$ | 0.0875 | 0.0750 | 0.0875 |

Give every probability to at least four significant figures.

**(a)** $p_1(0) =$ ______________

**(b)** $p_2(2) =$ ______________

**(c)** $p_1(0)\, p_2(2) =$ ______________

**(d)** Check the whole table against Theorem 5.4. Are $Y_1$ and $Y_2$ independent?

◯ Independent: every cell equals the product of its marginals
◯ Dependent: at least one cell differs from the product of its marginals

## Problem 8

seed 20260925

Two unrelated Azerbaijani exporters sign export contracts week by week: $Y_1$ is the number signed next week by a pomegranate-juice producer in Goychay, $Y_2$ the number signed by a carpet workshop in Guba. They sell different products to different buyers, and an analyst models $Y_1$ and $Y_2$ as independent with marginal probability functions

| $y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_1(y)$ | 0.13 | 0.33 | 0.21 | 0.33 |
| $p_2(y)$ | 0.32 | 0.32 | 0.19 | 0.17 |

Give every probability to at least four significant figures.

**(a)** $p(2, 1) =$ ______________

**(b)** $P(Y_1 = Y_2) =$ ______________

**(c)** The probability that the juice producer signs more contracts than the carpet workshop, $P(Y_1 > Y_2) =$

______________

**(d)** $P(Y_1 + Y_2 \geq 5) =$ ______________

## Problem 9

seed 20260925

A bank caps cash withdrawals at its branches and logs every customer who asks to exceed the cap. Let $Y_1$ be the number of such requests tomorrow at its Mingachevir branch and $Y_2$ the number at its Shirvan branch. The branches have separate customer bases and the bank models $Y_1$ and $Y_2$ as independent Poisson random variables with means $\lambda_1 = 1.1$ and $\lambda_2 = 2.9$.

Give every probability to at least four significant figures.

**(a)** $p(1, 3) = P(Y_1 = 1,\ Y_2 = 3) =$ ______________

**(b)** The probability of at most two requests across both branches, $P(Y_1 + Y_2 \leq 2) =$ ______________

**(c)** $P(Y_1 + Y_2 \leq 2,\ Y_1 > Y_2) =$ ______________

## Problem 10

seed 20260925

Each density below is a joint density for a pair $(Y_1, Y_2)$ (for instance, the weights of two assets in a randomly drawn client portfolio, or two waiting times), and is zero outside the region stated. Decide in each case whether $Y_1$ and $Y_2$ are independent, and why.

**(a)** $f(y_1, y_2) = y_1 + y_2, \quad 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1$

- ◯ Independent: the support is a rectangle and f factors as g(y1)h(y2)
- ◯ Dependent: the support is a rectangle but f does not factor
- ◯ Dependent: f may factor algebraically, but the support is not a rectangle

**(b)** $f(y_1, y_2) = 6y_1y_2^2, \quad 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1$

- ◯ Independent: the support is a rectangle and f factors as g(y1)h(y2)
- ◯ Dependent: the support is a rectangle but f does not factor
- ◯ Dependent: f may factor algebraically, but the support is not a rectangle

A pension fund splits each new client's contributions between equities ($Y_1$) and bonds ($Y_2$), with the rest in cash, so $Y_1 + Y_2 \le 1$. Across clients the weights have joint density $f(y_1, y_2) = 24y_1y_2$ for $y_1 \ge 0,\ y_2 \ge 0,\ y_1 + y_2 \le 1$, and zero elsewhere. The marginal density of each weight is $12y(1-y)^2$ on $[0, 1]$. Give each value to at least four significant figures.

**(c)** $f_1(0.60) =$ ______________

**(d)** $f_1(0.60)\, f_2(0.70) =$ ______________ (compare with $f(0.60, 0.70)$)

## Problem 11

seed 20260925

Two banks listed on the Baku Stock Exchange both lend heavily to firms in the oil-services sector, so news about oil prices tends to move their shares together. Classify each bank's daily share-price move as down (D), flat (F) or up (U), and let $Y_1$ be bank A's move and $Y_2$ bank B's. Over a long history the joint probability function is

| | $y_2 = \mathrm{D}$ | $y_2 = \mathrm{F}$ | $y_2 = \mathrm{U}$ |
|---|---|---|---|
| $y_1 = \mathrm{D}$ | 0.094 | 0.040 | 0.066 |
| $y_1 = \mathrm{F}$ | 0.090 | 0.060 | 0.150 |
| $y_1 = \mathrm{U}$ | 0.116 | 0.100 | 0.284 |

A risk analyst's spreadsheet assumes the two moves are independent. Give every number to at least four significant figures.

**(a)** The probability that both shares fall on the same day according to the independence assumption, $p_1(\mathrm{D})\, p_2(\mathrm{D}) =$ ______________

**(b)** The actual $P(Y_2 = \mathrm{D} \mid Y_1 = \mathrm{D}) =$ ______________

**(c)** The factor by which the independence assumption understates the probability of a joint fall, $\dfrac{p(\mathrm{D},\mathrm{D})}{p_1(\mathrm{D})\, p_2(\mathrm{D})} =$ ______________

## Problem 12

seed 20260925

An insurer studies claims arriving through one broker in Sumqayit. For a claim just filed, $Y_1$ is the number of days until it is settled and $Y_2$ the number of days until the broker files the next claim. The insurer's joint model is

$$f(y_1, y_2) = \begin{cases} c\, y_1\, e^{-y_1/2.5 - y_2/10}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** The constant $c =$ ______________ (give at least four significant figures)

**(b)** Use Theorem 5.5 to decide whether $Y_1$ and $Y_2$ are independent, and identify the marginal distribution of $Y_1$. Then find the probability that the claim is settled within $6$ days, $P(Y_1 \le 6) =$ ______________

**(c)** $P(Y_1 \le 6,\ Y_2 > 14) =$ ______________

## Problem 13

seed 20260925

Two telesales agents at a Baku bank cold-call prospects to sell a new credit card; each works through a separate list of prospects and keeps calling until the first sale. Agent A's calls succeed independently with probability $p_1 = 0.10$ each, agent B's with probability $p_2 = 0.28$. Let $Y_1$ and $Y_2$ be the numbers of calls the two agents need; $Y_1$ and $Y_2$ are independent geometric random variables.

Give every probability to at least four significant figures.

**(a)** The probability that both agents make their first sale on call number $3$, $P(Y_1 = 3,\ Y_2 = 3) =$ ______________

**(b)** The probability that both make their first sale on the same call number, $P(Y_1 = Y_2) =$ ______________

**(c)** The probability that agent A makes a sale strictly before agent B, $P(Y_1 < Y_2) =$ ______________

## Problem 14

seed 20260925

A bank has extended revolving credit lines of equal size to two unrelated corporate clients, a cement producer and a mobile-games studio. Let $Y_1$ and $Y_2$ be the fractions of their lines the two clients have drawn at month end. The bank models $Y_1$ and $Y_2$ as independent, each uniformly distributed on $(0, 1)$.

Give every probability to at least four significant figures.

**(a)** $P(Y_1 < 0.30\,Y_2) =$ ______________

**(b)** $P(Y_1 < 2.9\,Y_2) =$ ______________

**(c)** $P(Y_1 < 2.9\,Y_2 \mid Y_1 < 4.2\,Y_2) =$ ______________

## Problem 15

seed 20260925

A Baku importer pays a supplier in Europe through two correspondent banks in sequence. The delay at the first bank, $Y_1$ hours, and the delay at the second, $Y_2$ hours, are independent (the banks are unrelated and in different countries), and each is uniformly distributed on $(0, 5)$. The total delay is $Y_1 + Y_2$.

Give each probability to at least four significant figures.

**(a)** $P(Y_1 + Y_2 \le 4.25) =$ ______________

**(b)** $P(Y_1 + Y_2 \le 6.75) =$ ______________

## Problem 16

seed 20260925

A bank has branches in Sheki and in Quba. The total cash withdrawn at each branch tomorrow, $Y_1$ and $Y_2$ (thousand AZN), is exponentially distributed with mean $\beta = 28$, and because the two towns' customers are unrelated the bank models $Y_1$ and $Y_2$ as independent. The regional treasury has $t = 159.6$ thousand AZN of cash to allocate and compares two policies:

- **Split:** each branch holds $t/2 = 79.8$, and a branch is short if its own withdrawals exceed its own cash.

- **Pooled:** the cash sits in one armoured-van reserve that serves both branches, which is enough whenever $Y_1 + Y_2 \le t$.

Give every probability to at least four significant figures.

**(a)** Under the split policy, the probability that neither branch is short, $P(Y_1 \le 79.8,\ Y_2 \le 79.8) =$ ______________

**(b)** Under the pooled policy, $P(Y_1 + Y_2 \le 159.6) =$ ______________

**(c)** Given that the pooled reserve is enough, the probability that the split policy would also have been enough: $P(Y_1 \le 79.8,\ Y_2 \le 79.8 \mid Y_1 + Y_2 \le 159.6) =$ ______________

## Problem 17

seed 20260925

A Baku bank's treasury waits each morning for two incoming SWIFT payments from two unrelated correspondent banks. Measured from 9:00, the arrival time of the payment from correspondent A, $Y_1$ hours, is exponential with mean $\beta_1 = 1.75$, and that from correspondent B, $Y_2$ hours, is exponential with mean $\beta_2 = 4.75$. The two arrival times are independent.

Give every probability to at least four significant figures.

**(a)** The probability that B's payment arrives first, $P(Y_1 > Y_2) =$ ______________

**(b)** $P(Y_1 < 2.5\,Y_2) =$ ______________

**(c)** $P(Y_1 > Y_2 \mid Y_1 < 2.5\,Y_2) =$ ______________

## Problem 18

seed 20260925

A Baku exporter sends containers by rail to the Black Sea port of Poti. Beyond the scheduled transit time, a container suffers two extra delays: $Y_1$ days at customs on the border, uniformly distributed on $(0, 2)$, and $Y_2$ days waiting for a vessel slot at the port, uniformly distributed on $(0, 8)$. The border and the port are run by unrelated authorities and the exporter models $Y_1$ and $Y_2$ as independent. The total extra delay is $Y_1 + Y_2$.

Give every probability to at least four significant figures.

**(a)** $P(Y_1 + Y_2 \le 1.0) =$ ______________

**(b)** $P(Y_1 + Y_2 \le 6.8) =$ ______________

**(c)** $P(Y_1 + Y_2 \le 9.0) =$ ______________

## Problem 19

seed 20260925

A bank offers overdraft lines to small traders at Baku's Sadarak market. For a randomly chosen trader at month end, $Y_1$ is the fraction of the line that has been drawn and $Y_2$ the fraction of the line that is overdue. An overdue amount must have been drawn first, so $Y_2 \le Y_1$. The bank's model is

$$f(y_1, y_2) = \begin{cases} c\,y_2^3, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every value to at least four significant figures.

**(a)** $c =$ ______________

**(b)** The marginal density of $Y_1$ at $0.35$: $f_1(0.35) =$ ______________

**(c)** The marginal density of $Y_2$ at $0.65$: $f_2(0.65) =$ ______________

**(d)** $f_1(0.35)\, f_2(0.65) =$ ______________

Compare (d) with $f(0.35, 0.65)$ and decide whether the drawn and overdue fractions are independent.

## Problem 20

seed 20260925

A bank's SME loan applications pass through two departments. The credit-risk review takes $Y_1$ hours, exponential with mean $\beta_1 = 5$; the legal review takes $Y_2$ hours, exponential with mean $\beta_2 = 15$. The two teams work independently on different aspects of the file, so $Y_1$ and $Y_2$ are independent. The bank's service-level target is $t = 24$ hours.

Give every probability to at least four significant figures.

**(a)** If the reviews are done one after the other, the file is finished within target when $Y_1 + Y_2 \le 24$. Find $P(Y_1 + Y_2 \le 24) =$ ______________

**(b)** $P(Y_1 > Y_2) =$ ______________

**(c)** If instead the two reviews run in parallel, the file is finished within target when both are done by $24$ hours. Find $P(Y_1 \le 24,\ Y_2 \le 24) =$ ______________