Week 13, Problem Set 1 - worked solutions

Wackerly 5.4 - Independent Random Variables

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A bank flags every cash withdrawal above 20,000 AZN for a compliance check. Let Y_1 be the number of flagged withdrawals at its Ganja branch tomorrow and Y_2 the number at its Lankaran branch. The two branches serve separate customer bases in cities 300 km apart, and the compliance team models Y_1 and Y_2 as **independent** random variables with marginal probability functions

y	0	1	2
$p_1(y)$	0.43	0.29	0.28
$p_2(y)$	0.38	0.33	0.29

(no branch ever flags more than two). Give every probability to at least four significant figures.

(a) $p(0,0) = P(Y_1 = 0, Y_2 = 0) =$ ______________

(b) $p(2,1) =$ ______________

(c) The probability that exactly two withdrawals are flagged in total, $P(Y_1 + Y_2 = 2) =$ ______________

(d) The probability that the two branches flag the same number, $P(Y_1 = Y_2) =$ ______________

Because Y_1 and Y_2 are independent, Theorem 5.4 says the joint probability function is the product of the marginals at **every** pair: $p(y_1, y_2) = p_1(y_1)\, p_2(y_2)$. The marginals are all we need to build the whole joint table.

(a) $p(0,0) = p_1(0)\, p_2(0) = 0.43 \times 0.38 = 0.1634$.

(b) $p(2,1) = p_1(2)\, p_2(1) = 0.28 \times 0.33 = 0.0924$.

(c) The event $\{Y_1 + Y_2 = 2\}$ is the union of the disjoint cells $(0,2)$, $(1,1)$, $(2,0)$:

$$P(Y_1 + Y_2 = 2) = 0.43(0.29) + 0.29(0.33) + 0.28(0.38) = 0.1247 + 0.0957 + 0.1064 = 0.3268.$$

(d) The diagonal cells:

$$P(Y_1 = Y_2) = p(0,0) + p(1,1) + p(2,2) = 0.1634 + 0.0957 + 0.0812 = 0.3403.$$

Independence is what licenses multiplying. Had the branches shared a cause (say, a rumour about the bank spreading in both cities), the marginals alone would not determine the joint table.

Problem 2

seed 20260925

A bank's card division records two indicators for each credit-card customer: $Y_1 = 1$ if the customer paid late last month (0 otherwise) and $Y_2 = 1$ if the customer holds a savings deposit with the bank (0 otherwise). For a randomly chosen customer the joint probability function is

	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	0.3654	0.5046
$y_1 = 1$	0.0546	0.0754

Give every probability to at least four significant figures.

(a) $p_1(1) = P(Y_1 = 1) =$ ______________

(b) $p_2(1) = P(Y_2 = 1) =$ ______________

(c) $p_1(1)\, p_2(1) =$ ______________

(d) Compare (c) with $p(1, 1)$. Are Y_1 and Y_2 independent?

- ◯ Independent: p(y1, y2) = p1(y1) p2(y2) in every cell
- ◯ Dependent: the product rule fails in at least one cell

(a) Sum the row: $p_1(1) = p(1, 0) + p(1, 1) = 0.0546 + 0.0754 = 0.13$.

(b) Sum the column: $p_2(1) = p(0, 1) + p(1, 1) = 0.5046 + 0.0754 = 0.58$.

(c) $p_1(1)\, p_2(1) = 0.13 \times 0.58 = 0.0754$.

(d) $p(1, 1) = 0.0754 = 0.0754 = p_1(1)\, p_2(1)$. The cell equals the product. In a 2 x 2 table that is enough: the other three cells are fixed by the marginals, so they equal their products too, and Theorem 5.4 gives independence. Whether a customer paid late tells the bank nothing about whether the customer holds a deposit.

Notice that the marginals in (a) and (b) are the same whichever way (d) comes out: two tables with identical marginals can differ in whether the variables are independent. Only the joint table can settle it.

Problem 3

seed 20260925

An insurer runs two unrelated claim desks: motor claims at its Ganja office and health claims at its Sheki office. Starting at 9:00 tomorrow, let Y_1 be the time, in hours, until the next motor claim arrives in Ganja and Y_2 the time until the next health claim arrives in Sheki. The insurer treats Y_1 and Y_2 as independent, with Y_1 exponential with mean $\beta_1 = 2.5$ and Y_2 exponential with mean $\beta_2 = 9.5$.

Give every probability to at least four significant figures.

(a) $P(Y_1 > 2.5) =$ ______________

(b) The probability that neither desk has a claim by its threshold, $P(Y_1 > 2.5,\ Y_2 > 7.5) =$ ______________

(c) $P(Y_1 \le 2.5,\ Y_2 \le 7.5) =$ ______________

For an exponential variable with mean β, $P(Y > t) = \int_t^\infty \beta^{-1}e^{-y/\beta}\,dy = e^{-t/\beta}$.

(a) $P(Y_1 > 2.5) = e^{-2.5/2.5} = 0.36788$.

(b) The joint density of independent variables is the product of the marginal densities (Theorem 5.4), so the double integral over the rectangle $(t_a, \infty) \times (t_b, \infty)$ splits into a product of single integrals:

$$P(Y_1 > 2.5, Y_2 > 7.5) = \int_{2.5}^\infty \int_{7.5}^\infty f_1(y_1) f_2(y_2)\,dy_2\,dy_1 = e^{-2.5/2.5}\,e^{-7.5/9.5} = 0.36788 \times 0.45408 = 0.16705.$$

(c) The same argument on the rectangle $[0, t_a] \times [0, t_b]$:

$$P(Y_1 \le 2.5, Y_2 \le 7.5) = \left(1 - e^{-2.5/2.5}\right)\left(1 - e^{-7.5/9.5}\right) = 0.63212 \times 0.54592 = 0.34508.$$

This is exactly the property the book uses to motivate Definition 5.8: for independent variables, $P(a < Y_1 \le b,\ c < Y_2 \le d) = P(a < Y_1 \le b)\,P(c < Y_2 \le d)$.

Problem 4

seed 20260925

A mobile operator monitors two base-station towers in different regions, on separate power grids. Let Y_1 be the fraction of tomorrow that tower A spends at peak load and Y_2 the time, in days, until tower B's next outage. Their marginal distribution functions are

$$F_1(y_1) = y_1^2, \quad 0 \le y_1 \le 1, \qquad\qquad F_2(y_2) = 1 - e^{-y_2/29}, \quad y_2 \ge 0,$$

(with $F_1 = 0$ below 0 and 1 above 1, and $F_2 = 0$ for $y_2 < 0$), and the operator models Y_1 and Y_2 as independent in the sense of Definition 5.8.

Give every probability to at least four significant figures.

(a) The joint distribution function at the point $(0.70, 5)$: $F(0.70, 5) = P(Y_1 \le 0.70,\ Y_2 \le 5) =$ ______________

(b) $P(0.25 < Y_1 \le 0.70,\ 5 < Y_2 \le 16) =$ ______________

By Definition 5.8, independence means $F(y_1, y_2) = F_1(y_1)F_2(y_2)$ for every pair (y_1, y_2).

(a) $F(0.70, 5) = F_1(0.70)\,F_2(5) = (0.70)^2\left(1 - e^{-5/29}\right) = 0.49000 \times 0.15837 = 0.07760$.

(b) From Section 5.2, for $a < b$ and $c < d$, $P(a < Y_1 \le b, c < Y_2 \le d) = F(b, d) - F(b, c) - F(a, d) + F(a, c)$. With the product form each term is a product of marginals:

$$F(0.70, 16) - F(0.70, 5) - F(0.25, 16) + F(0.25, 5) = 0.20778 - 0.07760 - 0.02650 + 0.00990 = 0.11358.$$

The four terms factor as $[F_1(0.70) - F_1(0.25)][F_2(16) - F_2(5)]$, which is the shortcut:

$$(0.49000 - 0.06250)(0.42404 - 0.15837) = 0.42750 \times 0.26567 = 0.11358.$$

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Problem 5

seed 20260925

For a small manufacturing firm, let Y_1 be the fraction of its bank credit line drawn at the end of a quarter and Y_2 the fraction of its customer invoices paid on time during that quarter. A lender's model gives the joint density

$$f(y_1, y_2) = \begin{cases} c\, y_1\, y_2^4, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) The constant $c =$ ______________

Theorem 5.5 shows Y_1 and Y_2 are independent, which you may use below. Give each probability to at least four significant figures.

(b) $P(Y_1 \le 0.45) =$ ______________

(c) $P(Y_1 \le 0.45,\ Y_2 > 0.75) =$ ______________

(a) The density must integrate to 1 over the unit square:

$$\int_0^1 \int_0^1 c\, y_1 y_2^4\, dy_2\, dy_1 = c \cdot \frac{1}{2} \cdot \frac{1}{5} = 1 \quad \Rightarrow \quad c = 2 \times 5 = 10.$$

The density is positive exactly on the rectangle $0 \le y_1 \le 1,\ 0 \le y_2 \le 1$ and factors as $g(y_1)h(y_2)$ with $g(y_1) = 10\, y_1$ and $h(y_2) = y_2^4$, both nonnegative. Theorem 5.5 therefore gives independence, without computing any marginal. (The marginals are in fact $f_1(y_1) = 2y_1$ and $f_2(y_2) = 5y_2^4$, constant multiples of g and h.)

(b) $P(Y_1 \le 0.45) = \int_0^{0.45} 2y_1\, dy_1 = (0.45)^2 = 0.20250.$

(c) By independence the rectangle probability is a product:

$$P(Y_1 \le 0.45, Y_2 > 0.75) = (0.45)^2 \left[1 - (0.75)^5\right] = 0.20250 \times 0.76270 = 0.15445.$$

Problem 6

seed 20260925

A bank classifies each loan application by segment, $Y_1 = 0$ (retail) or $Y_1 = 1$ (SME), and by channel, $Y_2 = 1$ (mobile app), 2 (branch) or 3 (sales agent). A fraction $P(Y_1 = 1) = 0.25$ of applications come from SMEs, and the conditional probability functions of the channel given the segment are

	$y_2 = 1$	$y_2 = 2$	$y_2 = 3$
$p(y_2 \mid Y_1 = 0)$	0.58	0.23	0.19
$p(y_2 \mid Y_1 = 1)$	0.51	0.23	0.26

Give every probability to at least four significant figures.

(a) $p(1, 1) = P(Y_1 = 1, Y_2 = 1) =$ ______________

(b) $p_2(1) = P(Y_2 = 1) =$ ______________

(c) Are Y_1 and Y_2 independent?

○ Independent: the conditional distribution of Y2 is the same in both rows

○ Dependent: the conditional distribution of Y2 changes with Y1

(a) By the definition of a conditional probability function, $p(1,1) = p_1(1)\,p(1 \mid Y_1 = 1) = 0.25 \times 0.51 = 0.1275$.

(b) Sum the joint probabilities over y_1:

$$p_2(1) = p_1(0)\,p(1 \mid Y_1 = 0) + p_1(1)\,p(1 \mid Y_1 = 1) = 0.75 \times 0.58 + 0.25 \times 0.51 = 0.4350 + 0.1275 = 0.5625.$$

(c) Exercise 5.44 (the discrete version of Exercise 5.43): Y_1 and Y_2 are independent if and only if $p(y_2 \mid y_1) = p_2(y_2)$ for all y_2 and every y_1 with $p_1(y_1) > 0$. This is Theorem 5.4 rewritten, since $p(y_1, y_2) = p_1(y_1)p(y_2 \mid y_1)$.

The two rows differ (for example 0.51 versus 0.58 for the app), so p(y2 | y1) is not p2(y2) for every y1, and the variables are dependent: SME applicants use the mobile app less and agents more than retail applicants do.

Problem 7

seed 20260925

A credit bureau in Baku summarises its files on individual borrowers by Y_1, the number of late payments in the last year (0, 1, or 2 or more, coded 2), and Y_2, the number of credit products held (1, 2, or 3). For a randomly chosen file the joint probability function is

	$y_2 = 1$	$y_2 = 2$	$y_2 = 3$
$y_1 = 0$	0.1750	0.1500	0.1750
$y_1 = 1$	0.0875	0.0750	0.0875
$y_1 = 2$	0.0875	0.0750	0.0875

Give every probability to at least four significant figures.

(a) $p_1(0) =$ ______________

(b) $p_2(2) =$ ______________

(c) $p_1(0)\,p_2(2) =$ ______________

(d) Check the whole table against Theorem 5.4. Are Y_1 and Y_2 independent?

○ Independent: every cell equals the product of its marginals

○ Dependent: at least one cell differs from the product of its marginals

The row totals give p_1 and the column totals give p_2: $p_1(0), p_1(1), p_1(2) = 0.50, 0.25, 0.25$ and $p_2(1), p_2(2), p_2(3) = 0.35, 0.30, 0.35$.

(a) $p_1(0) = 0.1750 + 0.1500 + 0.1750 = 0.50$.

(b) $p_2(2) = 0.1500 + 0.0750 + 0.0750 = 0.30$.

(c) $p_1(0)\,p_2(2) = 0.50 \times 0.30 = 0.1500$.

(d) $p(0,2) = 0.1500 = 0.1500$. This cell passes, and so does every other: each of the nine entries is the product of its row and column totals (check, e.g., p(2,3) = 0.25 x 0.35 = 0.0875). Theorem 5.4 needs equality at all pairs, and it holds, so Y1 and Y2 are independent.

Problem 8

seed 20260925

Two unrelated Azerbaijani exporters sign export contracts week by week: Y_1 is the number signed next week by a pomegranate-juice producer in Goychay, Y_2 the number signed by a carpet workshop in Guba. They sell different products to different buyers, and an analyst models Y_1 and Y_2 as independent with marginal probability functions

y	0	1	2	3
$p_1(y)$	0.13	0.33	0.21	0.33
$p_2(y)$	0.32	0.32	0.19	0.17

Give every probability to at least four significant figures.

(a) $p(2,1) =$ ______________

(b) $P(Y_1 = Y_2) =$ ______________

(c) The probability that the juice producer signs more contracts than the carpet workshop, $P(Y_1 > Y_2) =$

____

(d) $P(Y_1 + Y_2 \geq 5) =$ ______________

By Theorem 5.4, independence means $p(y_1, y_2) = p_1(y_1)\,p_2(y_2)$ for every pair, so each of the 16 cells of the joint table is a product of marginals, and any event's probability is a sum of such products.

(a) $p(2,1) = p_1(2)\,p_2(1) = 0.21 \times 0.32 = 0.0672$.

(b) The diagonal:

$$P(Y_1 = Y_2) = \sum_{y=0}^{3} p_1(y)p_2(y) = 0.0416 + 0.1056 + 0.0399 + 0.0561 = 0.2432.$$

(c) Condition on the value of Y_1 and collect the cells below the diagonal row by row:

$$P(Y_1 > Y_2) = p_1(1)P(Y_2 \leq 0) + p_1(2)P(Y_2 \leq 1) + p_1(3)P(Y_2 \leq 2)$$

$$= 0.33(0.32) + 0.21(0.64) + 0.33(0.83) = 0.1056 + 0.1344 + 0.2739 = 0.5139.$$

(For a check, $P(Y_1 < Y_2) = 1 - 0.2432 - 0.5139 = 0.2429$.)

(d) The cells with $y_1 + y_2 \geq 5$ are $(2,3), (3,2), (3,3)$:

$$0.21(0.17) + 0.33(0.19) + 0.33(0.17) = 0.0357 + 0.0627 + 0.0561 = 0.1545.$$

Problem 9

seed 20260925

A bank caps cash withdrawals at its branches and logs every customer who asks to exceed the cap. Let Y_1 be the number of such requests tomorrow at its Mingachevir branch and Y_2 the number at its Shirvan branch. The branches have separate customer bases and the bank models Y_1 and Y_2 as independent Poisson random variables with means $\lambda_1 = 1.1$ and $\lambda_2 = 2.9$.

Give every probability to at least four significant figures.

(a) $p(1,3) = P(Y_1 = 1,\ Y_2 = 3) =$ ______________

(b) The probability of at most two requests across both branches, $P(Y_1 + Y_2 \le 2) =$ ______________

(c) $P(Y_1 + Y_2 \le 2,\ Y_1 > Y_2) =$ ______________

Each marginal is Poisson: $p_i(y) = \lambda_i^y e^{-\lambda_i}/y!$. By Theorem 5.4, independence means the joint probability function is

$$p(y_1, y_2) = \frac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!} \cdot \frac{\lambda_2^{y_2} e^{-\lambda_2}}{y_2!}, \qquad y_1, y_2 = 0, 1, 2, \ldots$$

(a) $p(1,3) = \frac{1.1^1 e^{-1.1}}{1} \cdot \frac{2.9^3 e^{-2.9}}{6} = 0.36616 \times 0.22366 = 0.08190$.

(b) The marginal values needed are $p_1(0), p_1(1), p_1(2) = 0.33287, 0.36616, 0.20139$ and $p_2(0), p_2(1), p_2(2) = 0.05502, 0.15957, 0.23137$. The event $\{Y_1 + Y_2 \le 2\}$ consists of six cells:

(y_1, y_2)	$p_1(y_1)p_2(y_2)$
$(0,0)$	0.01832
$(0,1)$	0.05312
$(1,0)$	0.02015
$(0,2)$	0.07702
$(1,1)$	0.05843
$(2,0)$	0.01108

and they sum to $P(Y_1 + Y_2 \le 2) = 0.23810$.

(c) Of those six cells, $y_1 > y_2$ holds only at $(1,0)$ and $(2,0)$: $0.02015 + 0.01108 = 0.03123$.

(So, given a quiet day with at most two requests in total, the chance that Mingachevir had more of them is $0.03123/0.23810 = 0.13115$.)

Problem 10

seed 20260925

Each density below is a joint density for a pair (Y_1, Y_2) (for instance, the weights of two assets in a randomly drawn client portfolio, or two waiting times), and is zero outside the region stated. Decide in each case whether Y_1 and Y_2 are independent, and why.

(a) $f(y_1, y_2) = y_1 + y_2, \quad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1$

○ Independent: the support is a rectangle and f factors as g(y1)h(y2)
○ Dependent: the support is a rectangle but f does not factor
○ Dependent: f may factor algebraically, but the support is not a rectangle

(b) $f(y_1, y_2) = 6y_1y_2^2, \quad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1$

○ Independent: the support is a rectangle and f factors as g(y1)h(y2)
○ Dependent: the support is a rectangle but f does not factor
○ Dependent: f may factor algebraically, but the support is not a rectangle

A pension fund splits each new client's contributions between equities (Y_1) and bonds (Y_2), with the rest in cash, so $Y_1 + Y_2 \le 1$. Across clients the weights have joint density $f(y_1, y_2) = 24y_1y_2$ for $y_1 \ge 0,\ y_2 \ge 0,\ y_1 + y_2 \le 1$, and zero elsewhere. The marginal density of each weight is $12y(1-y)^2$ on $[0, 1]$. Give each value to at least four significant figures.

(c) $f_1(0.60) =$ _______________

(d) $f_1(0.60)\, f_2(0.70) =$ _______________ (compare with $f(0.60, 0.70)$)

Theorem 5.5 applies only when the density is positive exactly on a rectangle $a \le y_1 \le b,\ c \le y_2 \le d$ (the ends may be infinite). On such a support, independence holds exactly when $f = g(y_1)h(y_2)$. If the support is not a rectangle the variables are dependent however f looks algebraically, because there are points (y_1, y_2) with $f_1(y_1) > 0$ and $f_2(y_2) > 0$ but $f(y_1, y_2) = 0$ (Example 5.14).

(a) The support is the unit square, but y1 + y2 cannot be written as g(y1)h(y2): if it could, then f(0,0) = g(0)h(0) = 0 would force g(0) = 0 or h(0) = 0, making f(0,y2) = 0 for all y2 or f(y1,0) = 0 for all y1, yet f(0,1) = f(1,0) = 1. So the variables are dependent (Exercise 5.60).

(b) The support is the unit square, a rectangle, and 6 y1 y2^2 = g(y1)h(y2) with g(y1) = 6y1, h(y2) = y2^2. Theorem 5.5 gives independence (this is Example 5.11).

(c) $f_1(0.60) = 12(0.60)(1-0.60)^2 = 12(0.60)(0.40)^2 = 1.15200$. (From $f_1(y_1) = \int_0^{1-y_1} 24y_1y_2\, dy_2 = 12y_1(1-y_1)^2$; note the upper limit depends on y_1.)

(d) $f_2(0.70) = 12(0.70)(0.30)^2 = 0.75600$, so $f_1(0.60)f_2(0.70) = 0.87091$. But $0.60 + 0.70 = 1.30 > 1$, so the point lies outside the triangle and $f(0.60, 0.70) = 0$. The product of the marginals is positive where the joint density is zero, so $f \ne f_1f_2$ and the weights are dependent, even though $24y_1y_2$ "factors". A client heavy in equities cannot also be heavy in bonds.

Problem 11

seed 20260925

Two banks listed on the Baku Stock Exchange both lend heavily to firms in the oil-services sector, so news about oil prices tends to move their shares together. Classify each bank's daily share-price move as down (D), flat (F) or up (U), and let Y_1 be bank A's move and Y_2 bank B's. Over a long history the joint probability function is

	$y_2 = \mathrm{D}$	$y_2 = \mathrm{F}$	$y_2 = \mathrm{U}$
$y_1 = \mathrm{D}$	0.094	0.040	0.066
$y_1 = \mathrm{F}$	0.090	0.060	0.150
$y_1 = \mathrm{U}$	0.116	0.100	0.284

A risk analyst's spreadsheet assumes the two moves are independent. Give every number to at least four significant figures.

(a) The probability that both shares fall on the same day according to the independence assumption, $p_1(\mathrm{D})\, p_2(\mathrm{D}) =$ ______________

(b) The actual $P(Y_2 = \mathrm{D} \mid Y_1 = \mathrm{D}) =$ ______________

(c) The factor by which the independence assumption understates the probability of a joint fall, $\dfrac{p(\mathrm{D},\mathrm{D})}{p_1(\mathrm{D})\, p_2(\mathrm{D})} =$ ______________

(a) From the first row and first column, $p_1(\mathrm{D}) = 0.094 + 0.040 + 0.066 = 0.2$ and $p_2(\mathrm{D}) = 0.094 + 0.090 + 0.116 = 0.3$, so $p_1(\mathrm{D})p_2(\mathrm{D}) = 0.2 \times 0.3 = 0.0600$.

(b) $P(Y_2 = \mathrm{D} \mid Y_1 = \mathrm{D}) = p(\mathrm{D},\mathrm{D})/p_1(\mathrm{D}) = 0.094/0.2 = 0.4700$, larger than the unconditional $p_2(\mathrm{D}) = 0.3$. A fall in bank A makes a fall in bank B more likely, so by Exercise 5.44 the moves are dependent.

(c) $0.094/0.0600 = 1.5667$. Because $p(\mathrm{D},\mathrm{D}) \neq p_1(\mathrm{D})p_2(\mathrm{D})$, Theorem 5.4 fails at this pair and Y_1, Y_2 are dependent. The shared exposure to oil is a common cause, exactly the situation in which independence must not be assumed.

The spreadsheet gets the marginal risks right (every row and column total is correct) but understates the chance that both holdings fall together by the factor 1.5667. A corollary: the actual probability that **at least one** of the two falls, $0.2 + 0.3 - 0.094 = 0.4060$, is **smaller** than the independent figure 0.4400. Co-movement concentrates losses on the same days, which is what makes diversification across the two banks weaker.

Problem 12

seed 20260925

An insurer studies claims arriving through one broker in Sumqayit. For a claim just filed, Y_1 is the number of days until it is settled and Y_2 the number of days until the broker files the next claim. The insurer's joint model is

$$f(y_1, y_2) = \begin{cases} c\, y_1\, e^{-y_1/2.5 - y_2/10}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) The constant $c =$ ______________ (give at least four significant figures)

(b) Use Theorem 5.5 to decide whether Y_1 and Y_2 are independent, and identify the marginal distribution of Y_1. Then find the probability that the claim is settled within 6 days, $P(Y_1 \le 6) =$ ______________

(c) $P(Y_1 \le 6,\ Y_2 > 14) =$ ______________

The density is positive exactly on $y_1 > 0,\ y_2 > 0$, a rectangle with infinite sides, and it factors as $g(y_1)h(y_2)$ with $g(y_1) = c\, y_1 e^{-y_1/2.5}$ and $h(y_2) = e^{-y_2/10}$. By Theorem 5.5, Y_1 and Y_2 are independent.

(a) The double integral splits:

$$1 = c\int_0^\infty y_1 e^{-y_1/2.5}\,dy_1 \int_0^\infty e^{-y_2/10}\,dy_2 = c\,\Gamma(2)(2.5)^2 \cdot 10 = c\,(62.50),$$

using $\int_0^\infty y^{\alpha-1}e^{-y/\beta}dy = \Gamma(\alpha)\beta^\alpha$. So $c = 1/62.50 = 0.016000$.

(b) Independent, as above. g is proportional to a gamma density with $\alpha = 2$, $\beta = 2.5$, and h to an exponential density with mean 10, so those are the marginals. For a gamma variable with $\alpha = 2$, integration by parts gives $P(Y_1 \le t) = 1 - e^{-t/\beta}(1 + t/\beta)$. With $t/\beta = 6/2.5 = 2.4000$,

$$P(Y_1 \le 6) = 1 - e^{-2.4000}(1 + 2.4000) = 0.69156.$$

(c) By independence,

$$P(Y_1 \le 6, Y_2 > 14) = P(Y_1 \le 6)\,P(Y_2 > 14) = 0.69156 \times e^{-14/10} = 0.69156 \times 0.24660 = 0.17054.$$

Note that g and h need not be densities themselves; Theorem 5.5 only needs the product form on a rectangle.

Problem 13

seed 20260925

Two telesales agents at a Baku bank cold-call prospects to sell a new credit card; each works through a separate list of prospects and keeps calling until the first sale. Agent A's calls succeed independently with probability $p_1 = 0.10$ each, agent B's with probability $p_2 = 0.28$. Let Y_1 and Y_2 be the numbers of calls the two agents need; Y_1 and Y_2 are independent geometric random variables.

Give every probability to at least four significant figures.

(a) The probability that both agents make their first sale on call number 3, $P(Y_1 = 3,\ Y_2 = 3) =$ ______________

(b) The probability that both make their first sale on the same call number, $P(Y_1 = Y_2) =$ ______________

(c) The probability that agent A makes a sale strictly before agent B, $P(Y_1 < Y_2) =$ ______________

Each marginal is geometric, $p_i(y) = q_i^{\,y-1}p_i$ for $y = 1, 2, \ldots$, with $q_1 = 0.90$, $q_2 = 0.72$. By Theorem 5.4 the joint probability function is $p(y_1, y_2) = q_1^{\,y_1-1}p_1\,q_2^{\,y_2-1}p_2$.

(a) $p(3,3) = (0.90)^2(0.10)\cdot(0.72)^2(0.28) = 0.011757$.

(b) Sum the diagonal. With $r = q_1q_2 = 0.6480$,

$$P(Y_1 = Y_2) = \sum_{y=1}^{\infty} p_1p_2\,(q_1q_2)^{y-1} = \frac{p_1p_2}{1 - q_1q_2} = \frac{0.0280}{0.3520} = 0.07955.$$

(c) $Y_1 < Y_2$ means that for some y, A succeeds on call y and B fails on each of calls $1, \ldots, y$: \[P(Y_1 y) = \sum_{y=1}^\infty q_1^{\,y-1}p_1\,q_2^{\,y} = \frac{p_1q_2}{1-q_1q_2} = \frac{0.0720}{0.3520} = 0.20455.\] Check: the symmetric formula gives $P(Y_1 > Y_2) = p_2q_1/(1 - q_1q_2) = 0.71591$, and $0.07955 + 0.20455 + 0.71591 = 1$, since $p_1p_2 + p_1q_2 + p_2q_1 = 1 - q_1q_2$.

Problem 14

seed 20260925

A bank has extended revolving credit lines of equal size to two unrelated corporate clients, a cement producer and a mobile-games studio. Let Y_1 and Y_2 be the fractions of their lines the two clients have drawn at month end. The bank models Y_1 and Y_2 as independent, each uniformly distributed on $(0, 1)$.

Give every probability to at least four significant figures.

(a) $P(Y_1 < 0.30\,Y_2) =$ ______________

(b) $P(Y_1 < 2.9\,Y_2) =$ ______________

(c) $P(Y_1 < 2.9\,Y_2 \mid Y_1 < 4.2\,Y_2) =$ ______________

By independence (Theorem 5.4) the joint density is $f(y_1, y_2) = f_1(y_1)f_2(y_2) = 1 \cdot 1 = 1$ on the unit square, so the probability of any region inside the square is its area.

(a) For $c = 0.30 < 1$ the region $y_1 < c\,y_2$ is the triangle with vertices $(0, 0)$, $(0, 1)$, $(c, 1)$; its area is $\frac{1}{2}c = \frac{1}{2}(0.30) = 0.1500$. Equivalently $\int_0^1 \int_0^{cy_2} dy_1\,dy_2 = c/2$.

(b) For $a = 2.9 > 1$ the line $y_1 = ay_2$ leaves the square through the right side at $y_2 = 1/a$. The complement $y_1 \geq ay_2$ is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1/a)$, of area $1/(2a)$. Hence

$$P(Y_1 < 2.9Y_2) = 1 - \frac{1}{2(2.9)} = 0.82759.$$

(c) Because $a < b$, $\{Y_1 < aY_2\} \subset \{Y_1 < bY_2\}$, so the intersection is just $\{Y_1 < aY_2\}$:

$$P(Y_1 < 2.9Y_2 \mid Y_1 < 4.2Y_2) = \frac{1 - 1/(2 \cdot 2.9)}{1 - 1/(2 \cdot 4.2)} = \frac{0.82759}{0.88095} = 0.93942.$$

Problem 15

seed 20260925

A Baku importer pays a supplier in Europe through two correspondent banks in sequence. The delay at the first bank, Y_1 hours, and the delay at the second, Y_2 hours, are independent (the banks are unrelated and in different countries), and each is uniformly distributed on $(0, 5)$. The total delay is $Y_1 + Y_2$.

Give each probability to at least four significant figures.

(a) $P(Y_1 + Y_2 \leq 4.25) =$ ______________

(b) $P(Y_1 + Y_2 \leq 6.75) =$ ______________

Write $\theta = 5$. By independence the joint density is the product of the marginals: $f(y_1, y_2) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$ on the square \(0

(a) $t_1 = 4.25 \leq 5$, so the region $y_1 + y_2 \leq t_1$ is the corner triangle with legs t_1:

$$P(Y_1 + Y_2 \leq t_1) = \int_0^{t_1} \int_0^{t_1 - y_1} \frac{1}{25}\,dy_2\,dy_1 = \frac{t_1^2}{2\,\theta^2} = \frac{(4.25)^2}{2(25)} = 0.36125.$$

(b) $t_2 = 6.75 > 5$, so the line $y_1 + y_2 = t_2$ cuts off the opposite corner. The complement $y_1 + y_2 > t_2$ is a triangle with legs $2\theta - t_2 = 10 - 6.75 = 3.25$:

$$P(Y_1 + Y_2 \le t_2) = 1 - \frac{(2\theta - t_2)^2}{2\,\theta^2} = 1 - \frac{(3.25)^2}{2(25)} = 0.78875.$$

The two formulas agree at $t = \theta$, where both give $1/2$: the distribution of the total is symmetric about θ.

Problem 16

seed 20260925

A bank has branches in Sheki and in Quba. The total cash withdrawn at each branch tomorrow, Y_1 and Y_2 (thousand AZN), is exponentially distributed with mean $\beta = 28$, and because the two towns' customers are unrelated the bank models Y_1 and Y_2 as independent. The regional treasury has $t = 159.6$ thousand AZN of cash to allocate and compares two policies:

- **Split:** each branch holds $t/2 = 79.8$, and a branch is short if its own withdrawals exceed its own cash.
- **Pooled:** the cash sits in one armoured-van reserve that serves both branches, which is enough whenever $Y_1 + Y_2 \le t$.

Give every probability to at least four significant figures.

(a) Under the split policy, the probability that neither branch is short, $P(Y_1 \le 79.8,\ Y_2 \le 79.8) =$ _______________

(b) Under the pooled policy, $P(Y_1 + Y_2 \le 159.6) =$ _______________

(c) Given that the pooled reserve is enough, the probability that the split policy would also have been enough: $P(Y_1 \le 79.8,\ Y_2 \le 79.8 \mid Y_1 + Y_2 \le 159.6) =$ _______________

By independence (Theorem 5.4) the joint density is $f(y_1, y_2) = \beta^{-2} e^{-(y_1+y_2)/\beta}$ for $y_1, y_2 > 0$.

(a) The event is a rectangle, so it factors:

$$P(Y_1 \le t/2,\, Y_2 \le t/2) = \left(1 - e^{-t/(2\beta)}\right)^2 = \left(1 - e^{-2.85}\right)^2 = (0.94216)^2 = 0.88766.$$

(b) The event $y_1 + y_2 \le t$ is a triangle, not a rectangle, so integrate:

$$P(Y_1 + Y_2 \le t) = \int_0^t \frac{1}{\beta} e^{-y_1/\beta} \left(1 - e^{-(t-y_1)/\beta}\right) dy_1 = \left(1 - e^{-t/\beta}\right) - \frac{t}{\beta} e^{-t/\beta} = 1 - e^{-t/\beta}\left(1 + \frac{t}{\beta}\right).$$

With $t/\beta = 5.70$: $1 - e^{-5.70}(1 + 5.70) = 0.97758$.

(c) If both branches are within $t/2$ then the total is within t, so the split event is a subset of the pooled event and $P(A \mid B) = P(A)/P(B) = 0.88766/0.97758 = 0.90801$.

Pooling wins: the same cash covers more days, because a heavy day in Sheki is usually offset by an ordinary one in Quba. That offsetting is exactly what independence buys; if the two branches' withdrawals rose and fell together, the pooled reserve would gain much less.

Problem 17

seed 20260925

A Baku bank's treasury waits each morning for two incoming SWIFT payments from two unrelated correspondent banks. Measured from 9:00, the arrival time of the payment from correspondent A, Y_1 hours, is exponential with mean $\beta_1 = 1.75$, and that from correspondent B, Y_2 hours, is exponential with mean $\beta_2 = 4.75$. The two arrival times are independent.

Give every probability to at least four significant figures.

(a) The probability that B's payment arrives first, $P(Y_1 > Y_2) =$ ______________

(b) $P(Y_1 < 2.5\,Y_2) =$ ______________

(c) $P(Y_1 > Y_2 \mid Y_1 < 2.5\,Y_2) =$ ______________

By independence the joint density is $f(y_1, y_2) = \frac{1}{\beta_1\beta_2} e^{-y_1/\beta_1 - y_2/\beta_2}$ for $y_1, y_2 > 0$. For any constant $k > 0$, integrate over y_1 first:

$$P(Y_1 < kY_2) = \int_0^\infty \frac{1}{\beta_2} e^{-y_2/\beta_2}\left(1 - e^{-ky_2/\beta_1}\right) dy_2 = 1 - \frac{1/\beta_2}{1/\beta_2 + k/\beta_1} = \frac{k\beta_2}{\beta_1 + k\beta_2}.$$

(a) With $k = 1$, $P(Y_1 < Y_2) = \beta_2/(\beta_1 + \beta_2) = 0.73077$, so

$$P(Y_1 > Y_2) = \frac{\beta_1}{\beta_1 + \beta_2} = \frac{1.75}{1.75 + 4.75} = 0.26923.$$

(b) With $k = 2.5$, $k\beta_2 = 11.875$:

$$P(Y_1 < 2.5Y_2) = \frac{11.875}{1.75 + 11.875} = 0.87156.$$

(c) The intersection $\{Y_1 > Y_2\} \cap \{Y_1 < 2.5Y_2\}$ is the wedge $Y_2 < Y_1 < 2.5Y_2$, whose probability is the difference of the two wedges from the origin:

$$P(Y_2 < Y_1 < 2.5Y_2) = P(Y_1 < 2.5Y_2) - P(Y_1 < Y_2) = 0.87156 - 0.73077 = 0.14079.$$

Therefore

$$P(Y_1 > Y_2 \mid Y_1 < 2.5Y_2) = \frac{0.14079}{0.87156} = 0.16154.$$

(With $\beta_1 = \beta_2 = 1$ and $k = 2$ this reduces to Exercise 5.63's answer, $(2/3 - 1/2)/(2/3) = 1/4$.)

Problem 18

seed 20260925

A Baku exporter sends containers by rail to the Black Sea port of Poti. Beyond the scheduled transit time, a container suffers two extra delays: Y_1 days at customs on the border, uniformly distributed on $(0, 2)$, and Y_2 days waiting for a vessel slot at the port, uniformly distributed on $(0, 8)$. The border and the port are run by unrelated authorities and the exporter models Y_1 and Y_2 as independent. The total extra delay is $Y_1 + Y_2$.

Give every probability to at least four significant figures.

(a) $P(Y_1 + Y_2 \le 1.0) =$ ______________

(b) $P(Y_1 + Y_2 \le 6.8) =$ ______________

(c) $P(Y_1 + Y_2 \le 9.0) =$ ______________

By Theorem 5.4 the joint density is the product of the marginals, $f(y_1, y_2) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ on the rectangle \(0

(a) $t = 1.0 < a$: the region is a corner triangle with legs t:

$$P = \frac{t^2/2}{ab} = \frac{(1.0)^2}{32} = 0.03125.$$

(b) $a \le t = 6.8 \le b$: for each $y_1 \in (0, a)$, y_2 runs from 0 to $t - y_1$, which stays below b. The region is a trapezoid:

$$P = \frac{1}{ab} \int_0^a (t - y_1)\, dy_1 = \frac{at - a^2/2}{ab} = \frac{t - a/2}{b} = \frac{6.8 - 1.0}{8} = 0.72500.$$

(c) $b < t = 9.0 < a + b = 10$: only the far corner triangle, with legs $a + b - t = 1.0$, lies above the line:

$$P = 1 - \frac{(a + b - t)^2/2}{ab} = 1 - \frac{(1.0)^2}{32} = 0.96875.$$

Each formula joins the next continuously: at $t = a$ both (a) and (b) give $a/(2b)$, and at $t = b$ both (b) and (c) give $1 - a/(2b)$.

Problem 19

seed 20260925

A bank offers overdraft lines to small traders at Baku's Sadarak market. For a randomly chosen trader at month end, Y_1 is the fraction of the line that has been drawn and Y_2 the fraction of the line that is overdue. An overdue amount must have been drawn first, so $Y_2 \le Y_1$. The bank's model is

$$f(y_1, y_2) = \begin{cases} c\, y_2^3, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give every value to at least four significant figures.

(a) $c =$ ______________

(b) The marginal density of Y_1 at 0.35: $f_1(0.35) =$ ______________

(c) The marginal density of Y_2 at 0.65: $f_2(0.65) =$ ______________

(d) $f_1(0.35)\, f_2(0.65) =$ ______________

Compare (d) with $f(0.35, 0.65)$ and decide whether the drawn and overdue fractions are independent.

(a) Integrate y_2 first, from 0 up to y_1:

$$\int_0^1 \int_0^{y_1} c\, y_1^0 y_2^3\, dy_2\, dy_1 = c \int_0^1 \frac{y_1^0\, y_1^4}{4}\, dy_1 = \frac{c}{4 \cdot 5} = 1 \quad \Rightarrow \quad c = 20.$$

(b) For $0 \le y_1 \le 1$, $f_1(y_1) = \int_0^{y_1} 20\, y_1^0 y_2^3\, dy_2 = \frac{20}{4} y_1^{0+4} = 5\, y_1^4$, so $f_1(0.35) = 5(0.35)^4 = 0.07503$.

(c) For $0 \le y_2 \le 1$, y_1 runs from y_2 up to 1 (a limit that depends on y_2):

$$f_2(y_2) = \int_{y_2}^{1} 20\, y_1^0 y_2^3\, dy_1 = \frac{20}{1}\, y_2^3\left(1 - y_2^1\right),$$

so $f_2(0.65) = 20.0000\,(0.65)^3\left(1-(0.65)^1\right) = 1.92238$.

(d) $f_1(0.35)f_2(0.65) = 0.07503 \times 1.92238 = 0.14424$.

But $0.65 > 0.35$, so the point $(y_1, y_2) = (0.35, 0.65)$ has more overdue than drawn and lies outside the support: $f(0.35, 0.65) = 0$. Since $f \neq f_1 f_2$ at this point (and on the whole region $y_2 > y_1$ inside the square, which has positive area), Y_1 and Y_2 are **dependent**. The formula $c\, y_1^0 y_2^3$ is a product of a function of y_1 and a function of y_2, but Theorem 5.5 needs the support to be a rectangle and a triangle is not one (compare Examples 5.12 and 5.14). The dependence is economic common sense: a trader who has drawn little cannot owe much.

Problem 20

seed 20260925

A bank's SME loan applications pass through two departments. The credit-risk review takes Y_1 hours, exponential with mean $\beta_1 = 5$; the legal review takes Y_2 hours, exponential with mean $\beta_2 = 15$. The two teams work independently on different aspects of the file, so Y_1 and Y_2 are independent. The bank's service-level target is $t = 24$ hours.

Give every probability to at least four significant figures.

(a) If the reviews are done one after the other, the file is finished within target when $Y_1 + Y_2 \leq 24$. Find $P(Y_1 + Y_2 \leq 24) =$ ______________

(b) $P(Y_1 > Y_2) =$ ______________

(c) If instead the two reviews run in parallel, the file is finished within target when both are done by 24 hours. Find $P(Y_1 \leq 24,\ Y_2 \leq 24) =$ ______________

By Theorem 5.4 the joint density is $f(y_1, y_2) = \frac{1}{\beta_1\beta_2} e^{-y_1/\beta_1 - y_2/\beta_2}$ for $y_1, y_2 > 0$.

(a) The event $y_1 + y_2 \leq t$ is a triangle. For fixed $y_1 \in (0, t)$, y_2 runs from 0 to $t - y_1$:

$$P(Y_1 + Y_2 \leq t) = \int_0^t \frac{1}{\beta_1} e^{-y_1/\beta_1}\left(1 - e^{-(t-y_1)/\beta_2}\right) dy_1 = \left(1 - e^{-t/\beta_1}\right) - \frac{e^{-t/\beta_2}}{\beta_1}\int_0^t e^{-y_1\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)} dy_1.$$

The last integral equals $\frac{\beta_1\beta_2}{\beta_2 - \beta_1}\left(1 - e^{-t/\beta_1 + t/\beta_2}\right)$ (here $\beta_1 \neq \beta_2$), and simplifying,

$$P(Y_1 + Y_2 \leq t) = 1 - \frac{\beta_2 e^{-t/\beta_2} - \beta_1 e^{-t/\beta_1}}{\beta_2 - \beta_1}.$$

With $e^{-t/\beta_1} = e^{-4.8000} = 0.00823$ and $e^{-t/\beta_2} = e^{-1.6000} = 0.20190$:

$$P(Y_1 + Y_2 \leq 24) = 1 - \frac{15(0.20190) - 5(0.00823)}{10} = 0.70127.$$

(b) Integrate $P(Y_2 < y_1) = 1 - e^{-y_1/\beta_2}$ against the density of Y_1:

$$P(Y_1 > Y_2) = \int_0^\infty \frac{1}{\beta_1} e^{-y_1/\beta_1}\left(1 - e^{-y_1/\beta_2}\right) dy_1 = 1 - \frac{\beta_2}{\beta_1 + \beta_2} = \frac{\beta_1}{\beta_1 + \beta_2} = \frac{5}{20} = 0.25000.$$

The slower legal team finishes first less than half the time, as it should.

(c) This event is a rectangle, so independence factors it:

$$P(Y_1 \leq 24, Y_2 \leq 24) = \left(1 - e^{-4.8000}\right)\left(1 - e^{-1.6000}\right) = 0.99177 \times 0.79810 = 0.79154.$$

Running the reviews in parallel roughly replaces the sum by the maximum, which is why (c) exceeds (a).

Answer key

Problem	Answers
1	(a) 0.1634, (b) 0.0924, (c) 0.3268, (d) 0.3403
2	(a) 0.13, (b) 0.58, (c) 0.0754, (d) Choice 1
3	(a) 0.367879, (b) 0.167048, (c) 0.345085
4	(a) 0.0776009, (b) 0.113576
5	(a) 10, (b) 0.2025, (c) 0.154446
6	(a) 0.1275, (b) 0.5625, (c) Dependent: ... with Y1
7	(a) 0.5, (b) 0.3, (c) 0.15, (d) Independent ... marginals
8	(a) 0.0672, (b) 0.2432, (c) 0.5139, (d) 0.1545
9	(a) 0.081895, (b) 0.238103, (c) 0.0312282
10	(a) Dependent ... factor, (b) Independent: ... y2), (c) 1.152, (d) 0.870912
11	(a) 0.06, (b) 0.47, (c) 1.56667
12	(a) 0.016, (b) 0.691559, (c) 0.170536
13	(a) 0.0117573, (b) 0.0795455, (c) 0.204545
14	(a) 0.15, (b) 0.827586, (c) 0.939422
15	(a) 0.36125, (b) 0.78875
16	(a) 0.887657, (b) 0.977582, (c) 0.908013
17	(a) 0.269231, (b) 0.87156, (c) 0.161538
18	(a) 0.03125, (b) 0.725, (c) 0.96875
19	(a) 20, (b) 0.0750313, (c) 1.92238, (d) 0.144238
20	(a) 0.70127, (b) 0.25, (c) 0.791535