

Week 13, Problem Set 2

Wackerly 5.5-5.7 - Expected Values of Functions of Random Variables and Covariance

This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku bank outsources the upkeep of its ATM network. In a given month let Y_1 be the number of cash-replenishment visits and Y_2 the number of breakdown repairs. The two counts are **not** independent (a busy month brings both more visits and more wear), but the bank knows

$$E(Y_1) = 38, \quad E(Y_2) = 4.9.$$

Contract 1 charges a fixed fee of 2000 AZN plus 90 AZN per visit and 190 AZN per repair, so the monthly cost is $C_1 = 2000 + 90Y_1 + 190Y_2$.

Contract 2 charges 155 AZN for every call-out of either kind and returns a rebate of 550 AZN each month: $C_2 = 155(Y_1 + Y_2) - 550$.

(a) $E(C_1) =$ _____ AZN

(b) $E(C_2) =$ _____ AZN

(c) The expected monthly saving from choosing contract 2, $E(C_1 - C_2) =$ _____ AZN (negative if contract 2 is dearer)

Problem 2

seed 20260925

An insurer in Baku sells a bundle of motor and home cover. For a randomly chosen bundled household let Y_1 be the number of motor claims and Y_2 the number of home claims filed in a year. From past data the joint probability function $p(y_1, y_2)$ is

$y_1 \backslash y_2$	0	1	2
0	0.71	0.07	0.05
1	0.07	0.08	0.02

(a) Use Definition 5.9 to find $E(Y_1 Y_2) =$ _____

(b) The expected total number of claims, $E(Y_1 + Y_2) =$ _____

(c) A motor claim costs the insurer 850 AZN on average and a home claim 2300 AZN, so the insurer's payout on the household is $g(Y_1, Y_2) = 850Y_1 + 2300Y_2$. Find $E[g(Y_1, Y_2)] =$ _____ AZN

Problem 3

seed 20260925

Let Y_1 and Y_2 be next month's returns, in percent, on the shares of two banks listed on the Baku Stock Exchange. An analyst's model gives

$$E(Y_1) = 0.8, \quad E(Y_2) = 1.7, \quad \sigma_1 = 5, \quad \sigma_2 = 8, \quad E(Y_1 Y_2) = 15.36.$$

Give each answer to at least four significant figures.

(a) $\text{Cov}(Y_1, Y_2) =$ _____ (percent squared)

(b) The correlation coefficient $\rho =$ _____

A colleague records the same returns as decimals, $W_1 = Y_1/100$ and $W_2 = Y_2/100$ (so a 1% return is 0.01).

(c) $\text{Cov}(W_1, W_2) =$ _____

(d) The correlation coefficient of W_1 and W_2 is _____

Problem 4

seed 20260925

A Zaqatala hazelnut exporter sells next season's crop at the world price. Let P be the price in AZN per kilogram and Q the volume shipped in tonnes. Price is set on world markets and the harvest by local weather, so P and Q are modelled as independent, with

$$E(P) = 6.6, \quad \sigma_P = 1.5, \quad E(Q) = 280, \quad \sigma_Q = 55.$$

Revenue in thousand AZN is $R = PQ$ (AZN per kg times tonnes).

(a) $E(R) =$ _____ thousand AZN

(b) $E(P^2) =$ _____

(c) $E(R^2) = E(P^2 Q^2) =$ _____

(d) $V(R) =$ _____ (thousand AZN squared)

Problem 5

seed 20260925

A commercial bank's loan book is worth 470 million AZN. Let Y_1 be the proportion of the book lent in foreign currency and Y_2 the proportion of the foreign-currency loans that have had to be restructured. The risk team's model for next year is the joint density

$$f(y_1, y_2) = \begin{cases} 10(1 - y_1)(1 - y_2)^4, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Because $Y_1 Y_2$ is the proportion of the **whole** book that is restructured foreign-currency lending, $BY_1 Y_2$ with $B = 470$ is its amount in million AZN.

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ _____

(b) $E(Y_2) =$ _____

(c) $E(Y_1 Y_2) =$ _____

(d) The expected amount of restructured foreign-currency loans, $E(470 Y_1 Y_2) =$ _____ million AZN

seed 20260925

Problem 6

A retail bank studies its clients. For a randomly chosen client let $Y_1 = 1$ if the client uses the bank's mobile app (0 otherwise) and $Y_2 = 1$ if the client took a consumer loan in the past year (0 otherwise). The joint probability function is

$y_1 \backslash y_2$	0	1
0	0.24	0.46
1	0.06	0.24

Give each numerical answer to at least four significant figures.

(a) $E(Y_1 Y_2) =$ _____

(b) $\text{Cov}(Y_1, Y_2) =$ _____

(c) The correlation coefficient $\rho =$ _____

(d) The sign of the covariance is

- ☐ Positive
☐ Negative
☐ Zero

(e) When a client both uses the app and has a consumer loan, the bank earns a commission of 36 AZN from the app's loan-servicing partner. The expected commission per client, $E(36 Y_1 Y_2)$, is _____ AZN

seed 20260925

Problem 7

Let Y_1 and Y_2 be next year's deposit growth, in percent, at two Azerbaijani banks. An analyst's report states

$$E(Y_1) = 6, \quad E(Y_2) = 9, \quad V(Y_1) = 9, \quad V(Y_2) = 81, \quad \text{Cov}(Y_1, Y_2) = 16.20.$$

Take the means and variances as correct.

(a) $\text{Cov}(Y_1, Y_1) =$ _____

(b) The correlation coefficient implied by the reported covariance, $\rho =$ _____ (at least four significant figures)

(c) Is the reported value of $\text{Cov}(Y_1, Y_2)$ possible?

- ☐ Yes, it is possible
☐ No, it is impossible

(d) The largest value $\text{Cov}(Y_1, Y_2)$ could take, given these variances: _____

seed 20260925

Problem 8

Y_1 and Y_2 are next month's returns, in percent, on the shares of a Baku oil-services company and a Baku construction company. Each return is modelled as low, middle or high, with joint probability function

$y_1 \backslash y_2$	-3	1	8
-2	0.15	0.08	0.07
1	0.07	0.33	0.02
8	0.02	0.08	0.18

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ _____

(b) $E(Y_2) =$ _____

(c) $E(Y_1 Y_2) =$ _____

(d) $\text{Cov}(Y_1, Y_2) =$ _____

seed 20260925

Problem 9

A Ganja dairy producer sells two product lines through the same supermarket chain. Let Y_1 be next month's revenue from yoghurt and Y_2 the revenue from cheese, both in thousand AZN. Its planning model uses the joint probability function

$y_1 \backslash y_2$	15	35	55
20	0.11	0.17	0.16
50	0.16	0.05	0.35

Give each answer to at least four significant figures.

(a) Expected total revenue, $E(Y_1 + Y_2) =$ _____ thousand AZN

(b) $E(Y_1 Y_2) =$ _____

(c) $\text{Cov}(Y_1, Y_2) =$ _____

(d) The correlation coefficient $\rho =$ _____

seed 20260925

Problem 10

A freight operator runs block trains from Baku to the Georgian border. Let Y_1 be the number of trains dispatched next week and Y_2 the fuel surcharge per train, in thousand AZN, which is set weekly and tends to be high when demand, and so the number of trains, is high. Their joint probability function is

$y_1 \backslash y_2$	2	8
1	0.17	0.08
2	0.22	0.13
3	0.03	0.37

The weekly cost, in thousand AZN, is a fixed 35, plus 19 per train, plus the surcharge on every train:

$$C = 35 + 19Y_1 + Y_1Y_2.$$

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ _____

(b) $E(Y_2) =$ _____

(c) $E(Y_1Y_2) =$ _____

(d) The expected weekly cost $E(C) =$ _____ thousand AZN

(e) A planner instead plugs the means into the cost formula, $35 + 19E(Y_1) + E(Y_1)E(Y_2)$. The amount by which this understates (or, if negative, overstates) $E(C)$ equals $\text{Cov}(Y_1, Y_2) =$ _____

Problem 11

seed 20260925

Two government bonds trade on the Baku Stock Exchange. Let Y_1 and Y_2 be tomorrow's price moves of the two bonds, in qepik per 1 AZN of face value, each of which is -2 , 0 or $+2$. A trader's model gives the joint probability function

$y_1 \backslash y_2$	-2	0	$+2$
-2	0.1000	0.0800	0.1000
0	0.0800	0.2800	0.0800
$+2$	0.1000	0.0800	0.1000

Give each numerical answer to at least four significant figures.

(a) $P(Y_1 = 2) =$ _____

(b) $\text{Cov}(Y_1, Y_2) =$ _____

(c) Are Y_1 and Y_2 independent?

- ☐ Independent
☐ Dependent

(d) Now consider the squared moves, which measure how much each bond's price changes regardless of direction. Find $\text{Cov}(Y_1^2, Y_2^2) =$ _____

Problem 12

seed 20260925

A Baku telecom operator's call centre pairs a senior and a junior agent on one shift. Let Y_1 and Y_2 be the proportions of the shift the two agents actually spend on calls. The operator's model for the pair is

$$f(y_1, y_2) = \begin{cases} c(2y_1 + 6y_2), & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

(a) The constant $c =$ _____

(b) $E(Y_1) =$ _____

(c) $E(Y_1 Y_2) =$ _____

(d) $\text{Cov}(Y_1, Y_2) =$ _____

(e) The value the pair generates on a shift is measured as $30Y_1 + 37Y_2$ AZN. Its expected value is _____ AZN

Problem 13

seed 20260925

A fuel distributor in Sumgait restocks a filling-station tank of capacity 75 thousand litres each Monday. Let Y_1 be the amount in stock at the start of the week and Y_2 the amount sold during the week, both in thousand litres. Sales cannot exceed stock, and the distributor's model is

$$f(y_1, y_2) = \begin{cases} \frac{2}{75^2}, & 0 \leq y_2 \leq y_1 \leq 75, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ _____ thousand litres

(b) $E(Y_2) =$ _____ thousand litres

(c) The expected amount left in the tank at the end of the week, $E(Y_1 - Y_2) =$ _____ thousand litres

(d) Each litre sold earns a margin of 0.13 AZN and each litre left over costs 0.05 AZN to hold and re-test. The expected weekly profit, $E\{1000(0.13 Y_2 - 0.05(Y_1 - Y_2))\}$, is _____ AZN

Problem 14

seed 20260925

When a small-business loan defaults, a Baku bank's loss is $L = Y_1(1 - Y_2)$, where Y_1 is the exposure at default (in hundred thousand AZN) and Y_2 is the proportion of it later recovered. The bank's model for a defaulted loan is the joint density

$$f(y_1, y_2) = \begin{cases} \frac{1}{1.2} e^{-y_1/1.2} \cdot 5 y_2^4, & y_1 > 0, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

- (a) The expected recovery rate $E(Y_2) =$ _____
- (b) The expected loss $E(L) =$ _____ hundred thousand AZN
- (c) $E(L^2) =$ _____
- (d) $V(L) =$ _____
- (e) The bank has 23 defaulted loans of this kind, each with the loss distribution above. Their expected total loss is _____ hundred thousand AZN

Problem 15

seed 20260925

A private-equity fund manager in Baku studies one quarter of fund activity. Let Y_1 be the proportion of the fund's capital committed to new deals and Y_2 the proportion actually drawn down from investors. Capital can only be drawn once it is committed, so $Y_2 \leq Y_1$. The manager's model is

$$f(y_1, y_2) = \begin{cases} k y_1 y_2^2, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

- (a) $k =$ _____
- (b) $E(Y_1) =$ _____
- (c) $E(Y_2) =$ _____
- (d) $E(Y_1 Y_2) =$ _____
- (e) $\text{Cov}(Y_1, Y_2) =$ _____
- (f) The quarter's management fee, in thousand AZN, is $90Y_1 + 280Y_2$. Its expected value is _____ thousand AZN

Problem 16

seed 20260925

An energy company owns a solar farm and a wind farm on the Absheron peninsula. Let Y_1 and Y_2 be tomorrow's capacity factors (output as a proportion of maximum) of the solar and the wind farm. The company's model is

$$f(y_1, y_2) = \begin{cases} 1 + (0.25)(1 - 2y_1)(1 - 2y_2), & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(Each marginal is uniform on $[0, 1]$, so $E(Y_1) = E(Y_2) = 1/2$ and $V(Y_1) = V(Y_2) = 1/12$.)

Give each numerical answer to at least four significant figures.

- (a) $E(Y_1 Y_2) =$ _____
- (b) $\text{Cov}(Y_1, Y_2) =$ _____

(c) $\rho =$ _____

(d) The sign of $\text{Cov}(Y_1, Y_2)$ is

- ☐ Positive
☐ Negative
☐ Zero

(e) Under a hybrid contract the company earns $35 Y_1 Y_2$ thousand AZN (the payment is scaled by both farms' output). Its expected value is _____ thousand AZN

Problem 17

seed 20260925

A reinsurer covers a Baku insurer's property and motor books. Let Y_1 and Y_2 be next quarter's claims (million AZN) from the property and the motor book. Its model makes (Y_1, Y_2) uniformly distributed over the triangle with vertices $(0, 0)$, $(0, 8.5)$ and $(3, 8.5)$, that is, over the region

$$0 \leq y_1 \leq 3, \frac{8.5}{3} y_1 \leq y_2 \leq 8.5.$$

(a) Before computing anything, look at the shape of the region. The sign of $\text{Cov}(Y_1, Y_2)$ is

- ☐ Positive
☐ Negative
☐ Zero

Give each numerical answer to at least four significant figures.

(b) $E(Y_1) =$ _____

(c) $E(Y_2) =$ _____

(d) $E(Y_1 Y_2) =$ _____

(e) $\text{Cov}(Y_1, Y_2) =$ _____

(f) $\rho =$ _____

Problem 18

seed 20260925

The treasury of a Baku bank splits its liquidity portfolio between government bonds, short-term central-bank notes and cash. Let Y_1 be the share held in bonds and Y_2 the share held in notes at the end of next month. The two shares cannot add to more than the whole portfolio, and the bank's asset-liability committee uses the model

$$f(y_1, y_2) = \begin{cases} c y_1 y_2^3, & y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

You may use, for non-negative integers p and q ,

$$\int_0^1 y^p (1-y)^q dy = \frac{p! q!}{(p+q+1)!}.$$

Give each answer to at least four significant figures.

(a) $c =$ _____

(b) $E(Y_1) =$ _____

(c) $E(Y_1 Y_2) =$ _____

(d) $\text{Cov}(Y_1, Y_2) =$ _____

(e) $\rho =$ _____

(f) The portfolio is worth 590 million AZN. The expected amount held in securities (bonds and notes together), $E(590(Y_1 + Y_2))$, is _____ million AZN

Problem 19

seed 20260925

A Central Bank examiner reviews a Baku bank's small-business portfolio. The pool contains 14 loans: 4 graded standard, 7 on the watch list and 3 non-performing. The examiner draws 4 loans at random without replacement. Let Y_1 be the number of watch-list loans and Y_2 the number of non-performing loans in the sample, so that

$$p(y_1, y_2) = \frac{\binom{7}{y_1} \binom{3}{y_2} \binom{4}{4-y_1-y_2}}{\binom{14}{4}}, \quad y_1, y_2 \geq 0, y_1 + y_2 \leq 4.$$

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ _____

(b) $E(Y_1 Y_2) =$ _____

(c) $\text{Cov}(Y_1, Y_2) =$ _____

(d) Reviewing costs 270 AZN per watch-list loan and 740 AZN per non-performing loan in the sample. In addition, every (watch-list, non-performing) pair in the sample must go to a joint committee at 520 AZN per pair, so the total cost is $C = 270Y_1 + 740Y_2 + 520Y_1 Y_2$. Find $E(C) =$ _____ AZN

Problem 20

seed 20260925

A solar plant near Baku reports its maintenance cost per unit of output. In a given month let Y_1 be the output (GWh) and Y_2 the maintenance bill (thousand AZN). Output depends on the weather and the bill on the service contract, and the joint density is

$$f(y_1, y_2) = \begin{cases} \frac{y_1^2 e^{-y_1/3.8}}{2(3.8)^3} \cdot \frac{1}{29} e^{-y_2/29}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

The reported figure is the unit cost Y_2/Y_1 , in thousand AZN per GWh. Give each answer to at least four significant figures.

(a) A manager estimates the unit cost as $E(Y_2)/E(Y_1)$. Find this value: _____

(b) $E(1/Y_1) =$ _____

(c) $E(Y_2/Y_1) =$ _____

(d) $E[(Y_2/Y_1)^2] =$ _____

(e) $V(Y_2/Y_1) =$ _____