Week 13, Problem Set 2 - worked solutions

Wackerly 5.5-5.7 - Expected Values of Functions of Random Variables and Covariance

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku bank outsources the upkeep of its ATM network. In a given month let Y_1 be the number of cash-replenishment visits and Y_2 the number of breakdown repairs. The two counts are **not** independent (a busy month brings both more visits and more wear), but the bank knows

$$E(Y_1) = 38, \qquad E(Y_2) = 4.9.$$

Contract 1 charges a fixed fee of 2000 AZN plus 90 AZN per visit and 190 AZN per repair, so the monthly cost is $C_1 = 2000 + 90Y_1 + 190Y_2$.

Contract 2 charges 155 AZN for every call-out of either kind and returns a rebate of 550 AZN each month: $C_2 = 155(Y_1 + Y_2) - 550$.

(a) $E(C_1) =$ ______________ AZN

(b) $E(C_2) =$ ______________ AZN

(c) The expected monthly saving from choosing contract 2, $E(C_1 - C_2) =$ ______________ AZN (negative if contract 2 is dearer)

Each cost is a sum of functions of (Y_1, Y_2), so Theorem 5.8 splits the expectation term by term, Theorem 5.7 takes constants outside, and Theorem 5.6 gives $E(c) = c$. None of this uses independence, which is why the dependence between visits and repairs does not matter here.

(a)

$$E(C_1) = E(2000) + 90E(Y_1) + 190E(Y_2) = 2000 + 90(38) + 190(4.9) = 2000 + 3420 + 931.0 = 6351.0.$$

(b)

$$E(C_2) = 155\,(E(Y_1) + E(Y_2)) - 550 = 155(42.9) - 550 = 6649.5 - 550 = 6099.5.$$

(c) By Theorem 5.8 again, $E(C_1 - C_2) = E(C_1) - E(C_2) = 6351.0 - 6099.5 = 251.5$ AZN.

The dependence between Y_1 and Y_2 would matter for the **spread** of the monthly bill, but the expected bill depends only on the two means.

Problem 2

seed 20260925

An insurer in Baku sells a bundle of motor and home cover. For a randomly chosen bundled household let Y_1 be the number of motor claims and Y_2 the number of home claims filed in a year. From past data the joint probability function $p(y_1, y_2)$ is

$y_1 \backslash y_2$	0	1	2
0	0.71	0.07	0.05
1	0.07	0.08	0.02

(a) Use Definition 5.9 to find $E(Y_1Y_2) =$ ______________

(b) The expected total number of claims, $E(Y_1 + Y_2) =$ ______________

(c) A motor claim costs the insurer 850 AZN on average and a home claim 2300 AZN, so the insurer's payout on the household is $g(Y_1, Y_2) = 850Y_1 + 2300Y_2$. Find $E[g(Y_1, Y_2)] =$ ______________ AZN

(a) Definition 5.9 with $g(y_1, y_2) = y_1y_2$ sums $y_1y_2\,p(y_1, y_2)$ over all six cells. Every cell with $y_1 = 0$ or $y_2 = 0$ contributes zero, leaving

$$E(Y_1Y_2) = (1)(1)(0.08) + (1)(2)(0.02) = 0.08 + 0.04 = 0.12.$$

(b) Theorem 5.8 gives $E(Y_1 + Y_2) = E(Y_1) + E(Y_2)$; each mean is found from the marginal of that variable (equivalently, Definition 5.9 with $g = y_1$ or $g = y_2$).

$p_1(1) = 0.07 + 0.08 + 0.02 = 0.17$, so $E(Y_1) = 0(1 - 0.17) + 1(0.17) = 0.17$.

$p_2(1) = 0.07 + 0.08 = 0.15$ and $p_2(2) = 0.05 + 0.02 = 0.07$, so $E(Y_2) = 1(0.15) + 2(0.07) = 0.29$.

Hence $E(Y_1 + Y_2) = 0.17 + 0.29 = 0.46$.

(c) By Theorems 5.7 and 5.8,

$$E[g(Y_1, Y_2)] = 850E(Y_1) + 2300E(Y_2) = 850(0.17) + 2300(0.29) = 811.50 \text{ AZN}.$$

This is the pure premium for the bundle: the amount the insurer must charge, before expenses and profit, just to cover expected claims.

Problem 3

seed 20260925

Let Y_1 and Y_2 be next month's returns, in percent, on the shares of two banks listed on the Baku Stock Exchange. An analyst's model gives

$$E(Y_1) = 0.8, \quad E(Y_2) = 1.7, \quad \sigma_1 = 5, \quad \sigma_2 = 8, \quad E(Y_1Y_2) = 15.36.$$

Give each answer to at least four significant figures.

(a) $\text{Cov}(Y_1, Y_2) =$ ______________ (percent squared)

(b) The correlation coefficient $\rho =$ ______________

A colleague records the same returns as decimals, $W_1 = Y_1/100$ and $W_2 = Y_2/100$ (so a 1% return is 0.01).

(c) $\text{Cov}(W_1, W_2) =$ ______________

(d) The correlation coefficient of W_1 and W_2 is ______________

(a) Theorem 5.10:

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - \mu_1\mu_2 = 15.36 - (0.8)(1.7) = 15.36 - 1.36 = 14.00.$$

(b)

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sigma_1\sigma_2} = \frac{14.00}{(5)(8)} = \frac{14.00}{40} = 0.3500.$$

(c) Go back to Definition 5.10. W_i has mean $\mu_i/100$, so $W_i - E(W_i) = (Y_i - \mu_i)/100$ and, by Theorem 5.7,

$$\text{Cov}(W_1, W_2) = E\left[\frac{(Y_1 - \mu_1)(Y_2 - \mu_2)}{100 \cdot 100}\right] = \frac{\text{Cov}(Y_1, Y_2)}{10000} = 0.001400.$$

(d) Each standard deviation also shrinks by the factor 100, so the ratio is unchanged:

$$\rho_W = \frac{\text{Cov}(Y_1, Y_2)/10000}{(\sigma_1/100)(\sigma_2/100)} = \rho = 0.3500.$$

This is exactly the point the book makes after Definition 5.10: the covariance depends on the units of measurement, so a value like 0.001400 cannot be called large or small at a glance, while ρ is unit-free and always lies in $[-1, 1]$.

Problem 4

seed 20260925

A Zaqatala hazelnut exporter sells next season's crop at the world price. Let P be the price in AZN per kilogram and Q the volume shipped in tonnes. Price is set on world markets and the harvest by local weather, so P and Q are modelled as independent, with

$$E(P) = 6.6, \quad \sigma_P = 1.5, \qquad E(Q) = 280, \quad \sigma_Q = 55.$$

Revenue in thousand AZN is $R = PQ$ (AZN per kg times tonnes).

(a) $E(R) =$ ______________ thousand AZN

(b) $E(P^2) =$ ______________

(c) $E(R^2) = E(P^2Q^2) =$ ______________

(d) $V(R) =$ ______________ (thousand AZN squared)

(a) P and Q are independent, so Theorem 5.9 with $g(P) = P$ and $h(Q) = Q$ gives

$$E(R) = E(PQ) = E(P)E(Q) = 6.6 \times 280 = 1848.0.$$

(b) From $V(P) = E(P^2) - [E(P)]^2$,

$$E(P^2) = \sigma_P^2 + \mu_P^2 = 2.25 + 43.56 = 45.81.$$

(c) Theorem 5.9 again, now with $g(P) = P^2$ and $h(Q) = Q^2$. First $E(Q^2) = 3025 + 78400 = 81425$, and then

$$E(P^2Q^2) = E(P^2)E(Q^2) = 45.81 \times 81425 = 3730079.2.$$

(d)

$$V(R) = E(R^2) - [E(R)]^2 = 3730079.2 - 3415104.0 = 314975.2,$$

so revenue has standard deviation about 561.23 thousand AZN.

Theorem 5.9 is what makes this short. Without independence neither $E(PQ)$ nor $E(P^2Q^2)$ could be computed from the two means and two variances alone: a correlation between price and volume would change both.

Problem 5

seed 20260925

A commercial bank's loan book is worth 470 million AZN. Let Y_1 be the proportion of the book lent in foreign currency and Y_2 the proportion of the foreign-currency loans that have had to be restructured. The risk team's model for next year is the joint density

$$f(y_1, y_2) = \begin{cases} 10\,(1-y_1)(1-y_2)^4, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Because Y_1Y_2 is the proportion of the **whole** book that is restructured foreign-currency lending, BY_1Y_2 with $B = 470$ is its amount in million AZN.

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ ______________

(b) $E(Y_2) =$ ______________

(c) $E(Y_1Y_2) =$ ______________

(d) The expected amount of restructured foreign-currency loans, $E(470\,Y_1Y_2) =$ ______________ million AZN

The density factors as $f(y_1, y_2) = f_1(y_1)f_2(y_2)$ with $f_1(y_1) = 2(1-y_1)$ and $f_2(y_2) = 5(1-y_2)^4$ on the unit square (each integrates to 1), so Y_1 and Y_2 are independent by Theorem 5.5.

(a) Substituting $u = 1 - y_1$,

$$E(Y_1) = \int_0^1 y_1\, 2(1-y_1)\, dy_1 = 2\int_0^1 (1-u)u\, du = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3} = 0.3333.$$

(b) In the same way $E(Y_2) = 1/6 = 0.1667$.

(c) By Theorem 5.9 with $g(Y_1) = Y_1$ and $h(Y_2) = Y_2$,

$$E(Y_1Y_2) = E(Y_1)E(Y_2) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18} = 0.0556.$$

The double integral of Definition 5.9 gives the same value; it simply splits into the product of the two single integrals above.

(d) By Theorem 5.7, $E(470\,Y_1Y_2) = 470\,E(Y_1Y_2) = 470/18 = 26.11$ million AZN.

Problem 6

seed 20260925

A retail bank studies its clients. For a randomly chosen client let $Y_1 = 1$ if the client uses the bank's mobile app (0 otherwise) and $Y_2 = 1$ if the client took a consumer loan in the past year (0 otherwise). The joint probability function is

$y_1 \backslash y_2$	0	1
0	0.24	0.46
1	0.06	0.24

Give each numerical answer to at least four significant figures.

(a) $E(Y_1Y_2) =$ ______________

(b) $\text{Cov}(Y_1, Y_2) =$ ______________

(c) The correlation coefficient $\rho =$ ______________

(d) The sign of the covariance is

- ○ Positive
- ○ Negative
- ○ Zero

(e) When a client both uses the app and has a consumer loan, the bank earns a commission of 36 AZN from the app's loan-servicing partner. The expected commission per client, $E(36\, Y_1Y_2)$, is ______________ AZN

(a) Y_1Y_2 is 1 only in the cell $(1, 1)$ and 0 elsewhere, so by Definition 5.9 $E(Y_1Y_2) = p(1, 1) = 0.24$.

(b) The marginals are $P(Y_1 = 1) = 0.06 + 0.24 = 0.3$ and $P(Y_2 = 1) = 0.46 + 0.24 = 0.7$; for a 0-1 variable the mean is the probability of a 1, so $\mu_1 = 0.3$ and $\mu_2 = 0.7$. Theorem 5.10:

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - \mu_1\mu_2 = 0.24 - (0.3)(0.7) = 0.24 - 0.21 = 0.03.$$

(c) A 0-1 variable with mean p has $E(Y^2) = p$ and variance $p - p^2 = p(1 - p)$, so $\sigma_1^2 = 0.21$ and $\sigma_2^2 = 0.21$.

$$\rho = \frac{0.03}{\sqrt{(0.21)(0.21)}} = 0.1429.$$

(d) The covariance is 0.03, so it is positive. The joint probability $p(1, 1)$ is more than the 0.21 that independence would require, meaning app users are more likely than other clients to take a consumer loan. Note that the sign is decided by one comparison, $p(1, 1)$ against $p_1(1)p_2(1)$.

(e) By Theorem 5.7 and part (a), $E(36\, Y_1Y_2) = 36(0.24) = 8.64$ AZN.

Problem 7

seed 20260925

Let Y_1 and Y_2 be next year's deposit growth, in percent, at two Azerbaijani banks. An analyst's report states

$$E(Y_1) = 6, \quad E(Y_2) = 9, \quad V(Y_1) = 9, \quad V(Y_2) = 81, \quad \text{Cov}(Y_1, Y_2) = 16.20.$$

Take the means and variances as correct.

(a) $\text{Cov}(Y_1, Y_1) =$ ______________

(b) The correlation coefficient implied by the reported covariance, $\rho =$ ______________ (at least four significant figures)

(c) Is the reported value of $\text{Cov}(Y_1, Y_2)$ possible?

- ○ Yes, it is possible
- ○ No, it is impossible

(d) The largest value $\text{Cov}(Y_1, Y_2)$ could take, given these variances: ______________

(a) In Definition 5.10 put $Y_2 = Y_1$: $\text{Cov}(Y_1, Y_1) = E[(Y_1 - \mu_1)^2] = V(Y_1) = 9$. A random variable's covariance with itself is its variance.

(b) $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{81} = 9$, so

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sigma_1\sigma_2} = \frac{16.20}{(3)(9)} = 0.6000.$$

(c) Every correlation coefficient satisfies $-1 \le \rho \le 1$. The value 0.6000 lies between -1 and 1, so nothing rules the reported covariance out.

(d) Since $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2$ and $\rho \le 1$, the largest possible covariance is $\sigma_1\sigma_2 = 27$. It is attained only when $\rho = 1$, that is, when the two growth rates lie exactly on a straight line with positive slope: one bank's growth is a fixed increasing linear function of the other's.

The means play no part in any of this; they were given only to tempt you to use them.

Problem 8

seed 20260925

Y_1 and Y_2 are next month's returns, in percent, on the shares of a Baku oil-services company and a Baku construction company. Each return is modelled as low, middle or high, with joint probability function

$y_1 \backslash y_2$	-3	1	8
-2	0.15	0.08	0.07
1	0.07	0.33	0.02
8	0.02	0.08	0.18

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ ______________

(b) $E(Y_2) =$ ______________

(c) $E(Y_1Y_2) =$ ______________

(d) $\text{Cov}(Y_1, Y_2) =$ ______________

(a) Row sums give the marginal of Y_1: $p_1(-2) = 0.30$, $p_1(1) = 0.42$, $p_1(8) = 0.28$, so

$$E(Y_1) = (-2)(0.30) + (1)(0.42) + (8)(0.28) = 2.06.$$

(b) Column sums give $p_2(-3) = 0.24$, $p_2(1) = 0.49$, $p_2(8) = 0.27$, so

$$E(Y_2) = (-3)(0.24) + (1)(0.49) + (8)(0.27) = 1.93.$$

(c) Definition 5.9 with $g(y_1, y_2) = y_1y_2$: multiply each cell's probability by the product of its row and column labels and add all nine terms. Row by row the products y_1y_2 are $(6, -2, -16)$, $(-3, 1, 8)$ and $(-24, 8, 64)$, so

$$\begin{aligned} E(Y_1Y_2) = &(6)(0.15) + (-2)(0.08) + (-16)(0.07) \\ &+(-3)(0.07) + (1)(0.33) + (8)(0.02) \\ &+(-24)(0.02) + (8)(0.08) + (64)(0.18) = 11.5800. \end{aligned}$$

(d) Theorem 5.10:

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - \mu_1\mu_2 = 11.5800 - (2.06)(1.93) = 11.5800 - 3.9758 = 7.6042.$$

The sign is positive: the two stocks tend to move together, which is typical of shares in one national market.

Problem 9

seed 20260925

A Ganja dairy producer sells two product lines through the same supermarket chain. Let Y_1 be next month's revenue from yoghurt and Y_2 the revenue from cheese, both in thousand AZN. Its planning model uses the joint probability function

$y_1 \backslash y_2$	15	35	55
20	0.11	0.17	0.16
50	0.16	0.05	0.35

Give each answer to at least four significant figures.

(a) Expected total revenue, $E(Y_1 + Y_2) =$ ____________ thousand AZN

(b) $E(Y_1 Y_2) =$ ____________

(c) $\text{Cov}(Y_1, Y_2) =$ ____________

(d) The correlation coefficient $\rho =$ ____________

(a) Marginals: $p_1(20) = 0.44$, $p_1(50) = 0.56$; $p_2(15) = 0.27$, $p_2(35) = 0.22$, $p_2(55) = 0.51$. Hence

$$\mu_1 = 20(0.44) + 50(0.56) = 36.80, \qquad \mu_2 = 15(0.27) + 35(0.22) + 55(0.51) = 39.80,$$

and by Theorem 5.8 $E(Y_1 + Y_2) = 36.80 + 39.80 = 76.60$.

(b) Definition 5.9 with $g = y_1 y_2$; the products in the six cells are $300, 700, 1100$ (first row) and $750, 1750, 2750$ (second row):

$$E(Y_1 Y_2) = 300(0.11) + 700(0.17) + 1100(0.16) + 750(0.16) + 1750(0.05) + 2750(0.35) = 1498.00.$$

(c) Theorem 5.10: $\text{Cov}(Y_1, Y_2) = 1498.00 - (36.80)(39.80) = 1498.00 - 1464.6400 = 33.3600$.

(d) The variances come from the marginals: $E(Y_1^2) = 1576.00$, so $\sigma_1^2 = 1576.00 - 36.80^2 = 221.7600$; $E(Y_2^2) = 1873.00$, so $\sigma_2^2 = 1873.00 - 39.80^2 = 288.9600$. Then

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sigma_1 \sigma_2} = \frac{33.3600}{\sqrt{(221.7600)(288.9600)}} = 0.1318.$$

The covariance is in (thousand AZN) squared and hard to judge on its own; ρ says how strong the linear relation between the two lines' revenues is on a scale from -1 to 1.

Problem 10

seed 20260925

A freight operator runs block trains from Baku to the Georgian border. Let Y_1 be the number of trains dispatched next week and Y_2 the fuel surcharge per train, in thousand AZN, which is set weekly and tends to be high when demand, and so the number of trains, is high. Their joint probability function is

$y_1 \backslash y_2$	2	8
1	0.17	0.08
2	0.22	0.13
3	0.03	0.37

The weekly cost, in thousand AZN, is a fixed 35, plus 19 per train, plus the surcharge on every train:

$$C = 35 + 19Y_1 + Y_1 Y_2.$$

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ ____________

(b) $E(Y_2) =$ ____________

(c) $E(Y_1 Y_2) =$ ____________

(d) The expected weekly cost $E(C) =$ ____________ thousand AZN

(e) A planner instead plugs the means into the cost formula, $35 + 19E(Y_1) + E(Y_1)E(Y_2)$. The amount by which this understates (or, if negative, overstates) $E(C)$ equals $\text{Cov}(Y_1, Y_2) =$ ____________

(a) Row sums: $p_1(1) = 0.25$, $p_1(2) = 0.35$, $p_1(3) = 0.40$, so $E(Y_1) = 1(0.25) + 2(0.35) + 3(0.40) = 2.15$.

(b) Column sums: $p_2(2) = 0.42$, $p_2(8) = 0.58$, so $E(Y_2) = 2(0.42) + 8(0.58) = 5.48$.

(c) Definition 5.9 with $g = y_1 y_2$. The cell products are $2, 8$ (one train), $4, 16$ (two), $6, 24$ (three):

$$E(Y_1 Y_2) = 2(0.17) + 8(0.08) + 4(0.22) + 16(0.13) + 6(0.03) + 24(0.37) = 13.00.$$

(d) Theorems 5.6, 5.7 and 5.8:

$$E(C) = 35 + 19E(Y_1) + E(Y_1Y_2) = 35 + 40.85 + 13.00 = 88.85.$$

(e) The planner's figure is $35 + 40.85 + (2.15)(5.48) = 87.6320$. The two differ only in the last term, so by Theorem 5.10

$$E(C) - \text{plug-in} = E(Y_1Y_2) - E(Y_1)E(Y_2) = 13.00 - 11.7820 = 1.2180 = \text{Cov}(Y_1, Y_2).$$

$E(Y_1Y_2) = E(Y_1)E(Y_2)$ is guaranteed only when the covariance is zero (for example under independence, Theorem 5.9). Here the surcharge is more likely to be high in busy weeks, the costly combination of many trains and a high surcharge carries extra weight, and plugging in the means misses that.

Problem 11

seed 20260925

Two government bonds trade on the Baku Stock Exchange. Let Y_1 and Y_2 be tomorrow's price moves of the two bonds, in qepik per 1 AZN of face value, each of which is -2, 0 or $+2$. A trader's model gives the joint probability function

$y_1 \backslash y_2$	-2	0	$+2$
-2	0.1000	0.0800	0.1000
0	0.0800	0.2800	0.0800
$+2$	0.1000	0.0800	0.1000

Give each numerical answer to at least four significant figures.

(a) $P(Y_1 = 2) =$ ______________

(b) $\text{Cov}(Y_1, Y_2) =$ ______________

(c) Are Y_1 and Y_2 independent?

- ○ Independent
- ○ Dependent

(d) Now consider the squared moves, which measure how much each bond's price changes regardless of direction. Find $\text{Cov}(Y_1^2, Y_2^2) =$

____

(a) Summing the row $y_1 = 2$: $P(Y_1 = 2) = 0.1000 + 0.0800 + 0.1000 = 0.2800$. By symmetry $P(Y_1 = -2) = 0.2800$ too, and $P(Y_1 = 0) = 2(0.0800) + 0.2800 = 0.4400$; Y_2 has the same marginal.

(b) The marginal is symmetric about 0, so $E(Y_1) = E(Y_2) = 0$. For $E(Y_1Y_2)$ (Definition 5.9) only the four corners have $y_1y_2 \neq 0$: two corners have product $+4$ and two have -4, all with the same probability 0.1000, so they cancel and $E(Y_1Y_2) = 0$. Theorem 5.10 gives $\text{Cov}(Y_1, Y_2) = 0 - 0 \cdot 0 = 0$.

(c) Independence requires $p(y_1, y_2) = p_1(y_1)p_2(y_2)$ in **every** cell. Here the centre cell is p(0,0) = 0.2800 while p1(0)p2(0) = 0.4400 x 0.4400 = 0.1936. One cell that fails the product rule is enough: Y1 and Y2 are dependent, although their covariance is zero. This is the converse of Theorem 5.11 failing, as in Example 5.24.

(d) Y_1^2 is 4 when the bond moves and 0 otherwise, so $E(Y_1^2) = 4 \times 2(0.2800) = 2.2400 = E(Y_2^2)$. $Y_1^2Y_2^2 = 16$ in the four corners and 0 elsewhere, so $E(Y_1^2Y_2^2) = 16 \times 0.4000 = 6.4000$. By Theorem 5.10,

$$\text{Cov}(Y_1^2, Y_2^2) = 6.4000 - (2.2400)^2 = 1.3824.$$

Zero covariance says only that there is no **linear** relation between the moves. When the table is not a product, the sizes of the moves can still be related, and the covariance of the squares picks that up.

Problem 12

seed 20260925

A Baku telecom operator's call centre pairs a senior and a junior agent on one shift. Let Y_1 and Y_2 be the proportions of the shift the two agents actually spend on calls. The operator's model for the pair is

$$f(y_1, y_2) = \begin{cases} c\,(2y_1 + 6y_2), & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

(a) The constant $c =$ ______________

(b) $E(Y_1) =$ ______________

(c) $E(Y_1Y_2) =$ ______________

(d) $\text{Cov}(Y_1, Y_2) =$ ______________

(e) The value the pair generates on a shift is measured as $30Y_1 + 37Y_2$ AZN. Its expected value is ______________ AZN

(a) $\int_0^1 \int_0^1 (2y_1 + 6y_2)\, dy_1\, dy_2 = 2/2 + 6/2 = 8/2$, so $c = 2/8 = 0.2500$.

(b) Definition 5.9 with $g = y_1$:

$$E(Y_1) = c\int_0^1 \int_0^1 y_1(2y_1 + 6y_2)\, dy_2\, dy_1 = c\int_0^1 \left(2y_1^2 + \frac{6}{2}y_1\right) dy_1 = c\left(\frac{2}{3} + \frac{6}{4}\right) = \frac{26}{48} = 0.5417.$$

In the same way $E(Y_2) = c(2/4 + 6/3) = 30/48 = 0.6250$.

(c)

$$E(Y_1Y_2) = c\int_0^1 \int_0^1 (2y_1^2 y_2 + 6y_1 y_2^2)\, dy_1\, dy_2 = c\left(\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{6}{3}\right) = \frac{2}{8}\cdot\frac{8}{6} = \frac{1}{3}.$$

(d) Theorem 5.10: $\text{Cov}(Y_1, Y_2) = \frac{1}{3} - (0.5417)(0.6250) = 0.333333 - 0.338542 = -0.005208$.

The covariance is negative, though small: the density puts extra weight where **one** of the two proportions is large, so a high value of one agent's time on calls goes slightly more often with a lower value for the other.

(e) Theorems 5.7 and 5.8: $E(30Y_1 + 37Y_2) = 30(0.5417) + 37(0.6250) = 39.38$ AZN.

Problem 13

seed 20260925

A fuel distributor in Sumgait restocks a filling-station tank of capacity 75 thousand litres each Monday. Let Y_1 be the amount in stock at the start of the week and Y_2 the amount sold during the week, both in thousand litres. Sales cannot exceed stock, and the distributor's model is

$$f(y_1, y_2) = \begin{cases} \frac{2}{75^2}, & 0 \le y_2 \le y_1 \le 75, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ ______________ thousand litres

(b) $E(Y_2) =$ ______________ thousand litres

(c) The expected amount left in the tank at the end of the week, $E(Y_1 - Y_2) =$ ______________ thousand litres

(d) Each litre sold earns a margin of 0.13 AZN and each litre left over costs 0.05 AZN to hold and re-test. The expected weekly profit, $E\{1000(0.13\,Y_2 - 0.05(Y_1 - Y_2))\}$, is ______________ AZN

For each fixed y_1, the inner variable y_2 runs from 0 to y_1.

(a) Definition 5.9 with $g = y_1$:

$$E(Y_1) = \int_0^{75} \int_0^{y_1} y_1 \frac{2}{75^2}\, dy_2\, dy_1 = \int_0^{75} \frac{2\,y_1^2}{75^2}\, dy_1 = \frac{2}{3}\cdot\frac{75^3}{75^2} = \frac{2(75)}{3} = 50.0000.$$

(b) With $g = y_2$ the inner integral is $\int_0^{y_1} y_2\, dy_2 = y_1^2/2$, so

$$E(Y_2) = \int_0^{75} \frac{y_1^2}{2}\cdot\frac{2}{75^2}\, dy_1 = \frac{1}{2}E(Y_1) = 25.0000.$$

(Given the stock, sales are spread evenly between 0 and the stock, so on average half the stock is sold.)

(c) Theorems 5.7 and 5.8 with $g_1 = Y_1$ and $g_2 = -Y_2$: $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 50.0000 - 25.0000 = 25.0000$.

(d) The same theorems give

$$1000\,(0.13\,E(Y_2) - 0.05\,E(Y_1 - Y_2)) = 1000(3.2500 - 1.2500) = 2000.00 \text{ AZN.}$$

The factor 1000 converts thousand litres to litres.

Problem 14

seed 20260925

When a small-business loan defaults, a Baku bank's loss is $L = Y_1(1 - Y_2)$, where Y_1 is the exposure at default (in hundred thousand AZN) and Y_2 is the proportion of it later recovered. The bank's model for a defaulted loan is the joint density

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{1.2} e^{-y_1/1.2} \cdot 5\,y_2^4, & y_1 > 0,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

(a) The expected recovery rate $E(Y_2) =$ ______________

(b) The expected loss $E(L) =$ ______________ hundred thousand AZN

(c) $E(L^2) =$ ______________

(d) $V(L) =$ ______________

(e) The bank has 23 defaulted loans of this kind, each with the loss distribution above. Their expected total loss is ______________ hundred thousand AZN

The density is a function of y_1 times a function of y_2 on a rectangular region, so by Theorem 5.5 Y_1 and Y_2 are independent: Y_1 is exponential with mean $\beta = 1.2$ (so $E(Y_1^2) = V(Y_1) + \beta^2 = 2\beta^2$), and Y_2 has density $f_2(y_2) = 5\,y_2^4$ on $[0, 1]$.

(a) $E(Y_2) = \int_0^1 y_2\,5\,y_2^4\,dy_2 = 5/6 = 0.8333$.

(b) By Theorem 5.9 with $g(Y_1) = Y_1$ and $h(Y_2) = 1 - Y_2$,

$$E(L) = E(Y_1)E(1 - Y_2) = 1.2\left(1 - \frac{5}{6}\right) = \frac{1.2}{6} = 0.2000.$$

(c) Theorem 5.9 again, with $g(Y_1) = Y_1^2$ and $h(Y_2) = (1 - Y_2)^2$. With $E(Y_2^2) = 5/7 = 0.7143$,

$$E[(1 - Y_2)^2] = 1 - 2E(Y_2) + E(Y_2^2) = 1 - \frac{2(5)}{6} + \frac{5}{7} = \frac{2}{42} = 0.0476,$$

so $E(L^2) = 2(1.2)^2 \times 0.0476 = 2.88 \times 0.0476 = 0.1371$.

(d) $V(L) = E(L^2) - [E(L)]^2 = 0.1371 - (0.2000)^2 = 0.0971$.

(e) The total loss is $L_1 + \cdots + L_{23}$, so Theorem 5.8 gives $23\,E(L) = 23(0.2000) = 4.6000$ hundred thousand AZN. This step needs no independence between the loans; the variance of the total would.

Problem 15

seed 20260925

A private-equity fund manager in Baku studies one quarter of fund activity. Let Y_1 be the proportion of the fund's capital committed to new deals and Y_2 the proportion actually drawn down from investors. Capital can only be drawn once it is committed, so $Y_2 \le Y_1$. The manager's model is

$$f(y_1, y_2) = \begin{cases} k\,y_1 y_2^2, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give each answer to at least four significant figures.

(a) $k =$ ______________

(b) $E(Y_1) =$ ______________

(c) $E(Y_2) =$ ______________

(d) $E(Y_1Y_2) =$ ______________

(e) $\text{Cov}(Y_1, Y_2) =$ ______________

(f) The quarter's management fee, in thousand AZN, is $90Y_1 + 280Y_2$. Its expected value is ______________ thousand AZN

Every expectation here has the form $E(Y_1^r Y_2^s)$. Integrate over y_2 first, from 0 to y_1, then over y_1 from 0 to 1:

$$\int_0^1 \int_0^{y_1} y_1^{1+r} y_2^{2+s}\, dy_2\, dy_1 = \int_0^1 \frac{y_1^{1+r+2+s+1}}{2+s+1}\, dy_1 = \frac{1}{(2+s+1)(1+2+r+s+2)}.$$

(a) With $r = s = 0$ the integral is $1/(3 \cdot 5)$, so $k = 3 \cdot 5 = 15$.

(b) $r = 1, s = 0$: $E(Y_1) = 15/(3 \cdot 6) = 15/18 = 0.8333$.

(c) $r = 0, s = 1$: $E(Y_2) = 15/(4 \cdot 6) = 15/24 = 0.6250$.

(d) $r = s = 1$: $E(Y_1Y_2) = 15/(4 \cdot 7) = 15/28 = 0.5357$.

(e) Theorem 5.10: $\text{Cov}(Y_1, Y_2) = 0.5357 - (0.8333)(0.6250) = 0.5357 - 0.520833 = 0.014881$.

The covariance is positive, as in Example 5.22: large drawdowns are possible only when commitments are large, because the support forces $y_2 \le y_1$.

(f) Theorems 5.7 and 5.8: $E(90Y_1 + 280Y_2) = 90(0.8333) + 280(0.6250) = 250.00$ thousand AZN.

Problem 16

seed 20260925

An energy company owns a solar farm and a wind farm on the Absheron peninsula. Let Y_1 and Y_2 be tomorrow's capacity factors (output as a proportion of maximum) of the solar and the wind farm. The company's model is

$$f(y_1, y_2) = \begin{cases} 1 + (0.25)(1 - 2y_1)(1 - 2y_2), & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(Each marginal is uniform on $[0, 1]$, so $E(Y_1) = E(Y_2) = 1/2$ and $V(Y_1) = V(Y_2) = 1/12$.)

Give each numerical answer to at least four significant figures.

(a) $E(Y_1Y_2) =$ ______________

(b) $\text{Cov}(Y_1, Y_2) =$ ______________

(c) $\rho =$ ______________

(d) The sign of $\text{Cov}(Y_1, Y_2)$ is

- ○ Positive
- ○ Negative
- ○ Zero

(e) Under a hybrid contract the company earns $35\, Y_1Y_2$ thousand AZN (the payment is scaled by both farms' output). Its expected value is ______________ thousand AZN

(a) Definition 5.9:

$$E(Y_1Y_2) = \int_0^1 \int_0^1 y_1 y_2\, dy_1\, dy_2 + (0.25) \int_0^1 \int_0^1 y_1(1 - 2y_1)\, y_2(1 - 2y_2)\, dy_1\, dy_2.$$

The first integral is $(1/2)(1/2) = 1/4$. The second splits into a product of two equal single integrals, $\int_0^1 y(1 - 2y)\, dy = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$, whose product is $1/36$. So

$$E(Y_1Y_2) = \frac{1}{4} + \frac{0.25}{36} = 0.256944.$$

(b) Theorem 5.10: $\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2) - \frac{1}{2} \cdot \frac{1}{2} = \frac{0.25}{36} = 0.006944$.

(c) $\sigma_1 = \sigma_2 = \sqrt{1/12}$, so $\rho = \dfrac{0.25/36}{1/12} = \dfrac{0.25}{3} = 0.0833$.

(d) The covariance has the sign of the dependence parameter, 0.25. It is positive: windy and sunny days tend to come together at this site, so high output at one farm goes with high output at the other.

Note that the marginals, and so the means and variances, do not depend on the parameter at all. Knowing the two marginal distributions tells you nothing about the covariance; that information lives only in the joint density.

(e) By Theorem 5.7, $E(35\,Y_1Y_2) = 35\,E(Y_1Y_2) = 35(0.256944) = 8.9931$ thousand AZN. If the farms were independent this would be $35/4$; the covariance shifts it by $35\,\mathrm{Cov}(Y_1, Y_2)$.

Problem 17

seed 20260925

A reinsurer covers a Baku insurer's property and motor books. Let Y_1 and Y_2 be next quarter's claims (million AZN) from the property and the motor book. Its model makes (Y_1, Y_2) uniformly distributed over the triangle with vertices (0, 0), (0, 8.5) and (3, 8.5), that is, over the region

$$0 \le y_1 \le 3, \ \tfrac{8.5}{3} y_1 \le y_2 \le 8.5.$$

(a) Before computing anything, look at the shape of the region. The sign of $\mathrm{Cov}(Y_1, Y_2)$ is

- ○ Positive
- ○ Negative
- ○ Zero

Give each numerical answer to at least four significant figures.

(b) $E(Y_1) =$ ____________

(c) $E(Y_2) =$ ____________

(d) $E(Y_1Y_2) =$ ____________

(e) $\mathrm{Cov}(Y_1, Y_2) =$ ____________

(f) $\rho =$ ____________

The triangle has area $\frac{1}{2}(3)(8.5) = 25.5/2$, so the uniform density is $f(y_1, y_2) = 2/25.5 = 0.0784$ on it. For each $y_1 \in [0, 3]$ the variable y_2 runs from $\frac{8.5}{3} y_1$ to 8.5.

(a) The triangle hugs the diagonal from the lower-left to the upper-right corner of its bounding rectangle: a large value of one claim total can occur only together with a large value of the other. The deviations from the means have the same sign more often than not, so the covariance is positive.

(b)-(d) Definition 5.9, integrating over y_2 first. Write $w = 3$ and $h = 8.5$. The inner integrals are $\int dy_2$ = (length of the y_2-range) and $\int y_2\,dy_2$ = (half the difference of the squared limits), which leaves

$$E(Y_1) = \frac{2}{wh} \int_0^w y_1 \cdot h \left(1 - \frac{y_1}{w}\right) dy_1 = \frac{2}{w} \left(\frac{w^2}{2} - \frac{w^2}{3}\right) = \frac{w}{3} = 1.0000,$$
$$E(Y_2) = \frac{2}{wh} \int_0^w \frac{h^2}{2} \left(1 - \frac{y_1^2}{w^2}\right) dy_1 = \frac{h}{w} \left(w - \frac{w}{3}\right) = \frac{2h}{3} = 5.6667,$$
$$E(Y_1Y_2) = \frac{2}{wh} \int_0^w y_1 \cdot \frac{h^2}{2} \left(1 - \frac{y_1^2}{w^2}\right) dy_1 = \frac{h}{w} \left(\frac{w^2}{2} - \frac{w^2}{4}\right) = \frac{wh}{4} = 6.3750.$$

As a check, the means are the coordinates of the triangle's centroid, the average of its three vertices.

(e) Theorem 5.10: $\text{Cov}(Y_1, Y_2) = 6.3750 - (1.0000)(5.6667) = 6.3750 - 5.6667 = 0.7083$, which is $+wh/36 = +25.5/36$, confirming (a).

(f) In the same way $E(Y_1^2)$ and $E(Y_2^2)$ give $V(Y_1) = 3^2/18 = 0.5000$ and $V(Y_2) = 8.5^2/18 = 4.0139$. Then

$$\rho = \frac{0.7083}{\sqrt{(0.5000)(4.0139)}} = 0.5000.$$

Every right triangle of this kind gives $\rho = \pm 1/2$, whatever its dimensions: rescaling either axis changes the covariance but not ρ.

Problem 18

seed 20260925

The treasury of a Baku bank splits its liquidity portfolio between government bonds, short-term central-bank notes and cash. Let Y_1 be the share held in bonds and Y_2 the share held in notes at the end of next month. The two shares cannot add to more than the whole portfolio, and the bank's asset-liability committee uses the model

$$f(y_1, y_2) = \begin{cases} c\, y_1 y_2^3, & y_1 \ge 0,\ y_2 \ge 0,\ y_1 + y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

You may use, for non-negative integers p and q,

$$\int_0^1 y^p (1-y)^q \, dy = \frac{p!\, q!}{(p+q+1)!}.$$

Give each answer to at least four significant figures.

(a) $c =$ ______________

(b) $E(Y_1) =$ ______________

(c) $E(Y_1 Y_2) =$ ______________

(d) $\text{Cov}(Y_1, Y_2) =$ ______________

(e) $\rho =$ ______________

(f) The portfolio is worth 590 million AZN. The expected amount held in securities (bonds and notes together), $E(590(Y_1 + Y_2))$, is ______________ million AZN

Every moment has the form $\iint y_1^P y_2^Q \, dy_2\, dy_1$ over the triangle. With y_2 from 0 to $1 - y_1$, the inner integral is $(1-y_1)^{Q+1}/(Q+1)$, and the given formula finishes the job:

$$\int_0^1 \int_0^{1-y_1} y_1^P y_2^Q \, dy_2\, dy_1 = \frac{1}{Q+1} \int_0^1 y_1^P (1-y_1)^{Q+1} dy_1 = \frac{P!\, Q!}{(P+Q+2)!}.$$

(a) With $P = 1$, $Q = 3$: $1 = c \cdot 1!\, 3!/6!$, so $c = 6!/(1!\, 3!) = 120$.

(b) $P = 2$, $Q = 3$:

$$E(Y_1) = \frac{6!}{1!\, 3!} \cdot \frac{2!\, 3!}{7!} = \frac{2}{7} = 0.2857.$$

Likewise $E(Y_2) = 4/7 = 0.5714$.

(c) $P = 2$, $Q = 4$:

$$E(Y_1 Y_2) = \frac{6!}{1!\, 3!} \cdot \frac{2!\, 4!}{8!} = \frac{2 \cdot 4}{7 \cdot 8} = \frac{8}{56} = 0.1429.$$

(d) Theorem 5.10: $\text{Cov}(Y_1, Y_2) = 0.1429 - (0.2857)(0.5714) = 0.1429 - 0.163265 = -0.020408$.

It is negative whatever the powers: the budget constraint $y_1 + y_2 \le 1$ means a month with a large bond share leaves less room for notes, and the two shares are pushed in opposite directions.

(e) $E(Y_1^2)$ uses $P = 3$, $Q = 3$ and equals $2 \cdot 3/(7 \cdot 8) = 0.1071$; similarly $E(Y_2^2) = 4 \cdot 5/(7 \cdot 8) = 0.3571$. So $V(Y_1) = 0.1071 - 0.2857^2 = 0.025510$ and $V(Y_2) = 0.3571 - 0.5714^2 = 0.030612$, and

$$\rho = \frac{-0.020408}{\sqrt{(0.025510)(0.030612)}} = -0.7303.$$

(f) Theorems 5.7 and 5.8: $E(590(Y_1+Y_2)) = 590(E(Y_1)+E(Y_2)) = 590(0.8571) = 505.71$ million AZN, and the remaining $590 - 505.71$ is expected to be held in cash.

Problem 19

seed 20260925

A Central Bank examiner reviews a Baku bank's small-business portfolio. The pool contains 14 loans: 4 graded standard, 7 on the watch list and 3 non-performing. The examiner draws 4 loans at random without replacement. Let Y_1 be the number of watch-list loans and Y_2 the number of non-performing loans in the sample, so that

$$p(y_1, y_2) = \frac{\binom{7}{y_1}\binom{3}{y_2}\binom{4}{4-y_1-y_2}}{\binom{14}{4}}, \qquad y_1, y_2 \geq 0,\ y_1 + y_2 \leq 4.$$

Give each answer to at least four significant figures.

(a) $E(Y_1) =$ ______________

(b) $E(Y_1Y_2) =$ ______________

(c) $\text{Cov}(Y_1, Y_2) =$ ______________

(d) Reviewing costs 270 AZN per watch-list loan and 740 AZN per non-performing loan in the sample. In addition, every (watch-list, non-performing) pair in the sample must go to a joint committee at 520 AZN per pair, so the total cost is $C = 270Y_1 + 740Y_2 + 520Y_1Y_2$. Find $E(C) =$ ______________ AZN

The denominator is $\binom{14}{4} = 1001$.

(a) The marginal of Y_1 is hypergeometric: the 4 loans are drawn from 7 watch-list loans and $14 - 7$ others. By Theorem 3.10, $E(Y_1) = 4(7)/14 = 2.0000$. (In the same way $E(Y_2) = 4(3)/14 = 0.8571$.) Summing $y_1\, p(y_1, y_2)$ over the table of Definition 5.9 gives the same value.

(b) Definition 5.9 with $g = y_1y_2$. Only cells with $y_1 \geq 1$ and $y_2 \geq 1$ contribute. With a sample of 4 these cells and their probabilities are

- $p(1,1) = \binom{7}{1}\binom{3}{1}\binom{4}{2}/1001 = 126/1001 = 0.125874$,
- $p(2,1) = \binom{7}{2}\binom{3}{1}\binom{4}{1}/1001 = 252/1001 = 0.251748$,
- $p(1,2) = \binom{7}{1}\binom{3}{2}\binom{4}{1}/1001 = 84/1001 = 0.083916$,

Because n = 4, the cells (2,2), (3,1) and (1,3) also contribute, with probabilities 0.062937, 0.104895 and 0.006993 (a probability is 0 when the pool has too few loans of a grade). Then

$$E(Y_1Y_2) = \sum_{y_1}\sum_{y_2} y_1y_2\, p(y_1, y_2) = 1(0.125874) + 2(0.251748) + 2(0.083916) + 4(0.062937) + 3(0.104895) + 3(0.006993) = 1.3846.$$

(c) Theorem 5.10: $\text{Cov}(Y_1, Y_2) = 1.3846 - (2.0000)(0.8571) = 1.3846 - 1.714286 = -0.329670$.

The covariance is negative: the sample has a fixed size, so every watch-list loan drawn takes a place that a non-performing loan could have occupied.

(d) Theorems 5.7 and 5.8:

$$E(C) = 270E(Y_1) + 740E(Y_2) + 520E(Y_1Y_2) = 270(2.0000) + 740(0.8571) + 520(1.3846) = 1894.29.$$

Note that the last term needs $E(Y_1Y_2)$, not $E(Y_1)E(Y_2)$; the two differ by exactly the covariance found in (c).

Problem 20

seed 20260925

A solar plant near Baku reports its maintenance cost per unit of output. In a given month let Y_1 be the output (GWh) and Y_2 the maintenance bill (thousand AZN). Output depends on the weather and the bill on the service contract, and the joint density is

$$f(y_1, y_2) = \begin{cases} \dfrac{y_1^2\, e^{-y_1/3.8}}{2\,(3.8)^3} \cdot \dfrac{1}{29} e^{-y_2/29}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

The reported figure is the unit cost Y_2/Y_1, in thousand AZN per GWh. Give each answer to at least four significant figures.

(a) A manager estimates the unit cost as $E(Y_2)/E(Y_1)$. Find this value: ______________

(b) $E(1/Y_1) =$ ______________

(c) $E(Y_2/Y_1) =$ ______________

(d) $E[(Y_2/Y_1)^2] =$ ______________

(e) $V(Y_2/Y_1) =$ ______________

The density is a function of y_1 alone times a function of y_2 alone on a product region, so Y_1 and Y_2 are independent by Theorem 5.5. Y_1 has a gamma distribution with $\alpha = 3$ and $\beta = 3.8$, and Y_2 is exponential with mean 29, so $E(Y_2^2) = 2(29)^2 = 1682$.

(a) $E(Y_1) = \alpha\beta = 11.4$, so $E(Y_2)/E(Y_1) = 29/11.4 = 2.5439$.

(b) Integrate against the gamma density; the power of y_1 drops by one and what is left is another gamma integral, $\int_0^\infty y^{\alpha-1} e^{-y/\beta} dy = \Gamma(\alpha)\beta^\alpha$:

$$E(1/Y_1) = \int_0^\infty \frac{1}{y_1} \cdot \frac{y_1^{\alpha-1} e^{-y_1/\beta}}{\Gamma(\alpha)\beta^\alpha} dy_1 = \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\beta(\alpha-1)} = \frac{1}{3.8(2)} = 0.131579.$$

(c) Theorem 5.9 with $g(Y_2) = Y_2$ and $h(Y_1) = 1/Y_1$:

$$E(Y_2/Y_1) = E(Y_2)\, E(1/Y_1) = \frac{29}{7.6} = 3.8158.$$

This is larger than the manager's 2.5439. $E(1/Y_1)$ is **not** $1/E(Y_1)$: low-output months push the unit cost up by more than high-output months pull it down.

(d) In the same way $E(1/Y_1^2) = \Gamma(\alpha-2)\beta^{\alpha-2}/(\Gamma(\alpha)\beta^\alpha) = 1/(\beta^2(\alpha-1)(\alpha-2)) = 0.034626$ (this needs $\alpha > 2$), and Theorem 5.9 with $g(Y_2) = Y_2^2$, $h(Y_1) = 1/Y_1^2$ gives

$$E[(Y_2/Y_1)^2] = E(Y_2^2)\, E(1/Y_1^2) = 1682 \times 0.034626 = 58.2410.$$

(e) $V(Y_2/Y_1) = 58.2410 - (3.8158)^2 = 43.6807$.

Answer key

Problem	Answers
1	(a) 6351, (b) 6099.5, (c) 251.5
2	(a) 0.12, (b) 0.46, (c) 811.5
3	(a) 14, (b) 0.35, (c) 0.0014, (d) 0.35
4	(a) 1848, (b) 45.81, (c) 3.73008E+06, (d) 314975
5	(a) 0.333333, (b) 0.166667, (c) 0.0555556, (d) 26.1111
6	(a) 0.24, (b) 0.03, (c) 0.142857, (d) Positive, (e) 8.64
7	(a) 9, (b) 0.6, (c) Yes, it is possible, (d) 27
8	(a) 2.06, (b) 1.93, (c) 11.58, (d) 7.6042
9	(a) 76.6, (b) 1498, (c) 33.36, (d) 0.131785
10	(a) 2.15, (b) 5.48, (c) 13, (d) 88.85, (e) 1.218
11	(a) 0.28, (b) 0, (c) Dependent, (d) 1.3824
12	(a) 0.25, (b) 0.541667, (c) 0.333333, (d) -0.00520833, (e) 39.375
13	(a) 50, (b) 25, (c) 25, (d) 2000
14	(a) 0.833333, (b) 0.2, (c) 0.137143, (d) 0.0971429, (e) 4.6
15	(a) 15, (b) 0.833333, (c) 0.625, (d) 0.535714, (e) 0.014881, (f) 250

16	(a) 0.256944, (b) 0.00694444, (c) 0.0833333, (d) Positive, (e) 8.99306
17	(a) Positive, (b) 1, (c) 5.66667, (d) 6.375, (e) 0.708333, (f) 0.5
18	(a) 120, (b) 0.285714, (c) 0.142857, (d) -0.0204082, (e) -0.730297, (f) 505.714
19	(a) 2, (b) 1.38462, (c) -0.32967, (d) 1894.29
20	(a) 2.54386, (b) 0.131579, (c) 3.81579, (d) 58.241, (e) 43.6807