

Week 14, Problem Set 1

Wackerly 5.8 - The Expected Value and Variance of Linear Functions; Portfolio Variance

This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

An investor in Baku splits her savings between shares of a listed bank and an AZN government-bond fund, with a fraction $a_1 = 0.30$ in the bank share and $a_2 = 0.70$ in the bond fund. Let Y_1 and Y_2 be the two monthly returns, in percent. Her analyst estimates

$$E(Y_1) = 2.2, \quad E(Y_2) = 0.6, \quad \sigma_1 = 11.5, \quad \sigma_2 = 4.0, \quad \rho = 0.60.$$

The portfolio return is $U = a_1Y_1 + a_2Y_2$. Give each answer to at least four significant figures.

(a) $E(U) =$ _____ (% per month)

(b) $V(U) =$ _____ ($\%^2$)

(c) The standard deviation of U is _____ (% per month)

Problem 2

seed 20260925

A construction contractor in Sumgait buys cement at 110 AZN per tonne and steel rebar at 1450 AZN per tonne. Let Y_1 and Y_2 be the tonnes of cement and of rebar bought in a week, with

$$E(Y_1) = 42, \quad V(Y_1) = 60, \quad E(Y_2) = 8, \quad V(Y_2) = 4.0.$$

The weekly spending is $C = 110Y_1 + 1450Y_2$ AZN. Give each answer to at least four significant figures.

(a) $E(C) =$ _____ AZN

A junior analyst treats Y_1 and Y_2 as independent.

(b) Under that assumption, $V(C) =$ _____ (AZN^2)

(c) and the standard deviation of C is _____ AZN.

(d) Both purchases rise and fall with the number of building sites that are active that week, and the records show a correlation $\rho = 0.6$ between Y_1 and Y_2 . With this correlation, $V(C) =$ _____ (AZN^2)

Problem 3

seed 20260925

A restaurant on Nizami Street records its daily revenue Y_1 and its daily food cost Y_2 , both in AZN. The owner's figures are

$$E(Y_1) = 2600, \quad V(Y_1) = 250000, \quad E(Y_2) = 1100, \quad V(Y_2) = 44100, \quad \text{Cov}(Y_1, Y_2) = 57750.$$

Rent and wages add a fixed **600** AZN a day, so the daily operating profit is $U = Y_1 - Y_2 - 600$. Give each answer to at least four significant figures.

(a) $E(U) =$ _____ AZN

(b) $V(U) =$ _____ (AZN²)

(c) The standard deviation of the daily profit is _____ AZN.

(d) For comparison, the variance of the total cash flowing through the till and the kitchen, $V(Y_1 + Y_2) =$

Problem 4

seed 20260925

The amount of a single cash withdrawal at a bank's ATMs in Ganja has mean $\mu = 100$ AZN and standard deviation $\sigma = 115$ AZN. An auditor draws $n = 55$ withdrawals at random from the log and treats their amounts $Y_1, \dots, Y_n$ as independent. Let $\bar{Y}$ be the sample mean. Give each answer to at least four significant figures.

(a) $E(\bar{Y}) =$ _____ AZN

(b) $V(\bar{Y}) =$ _____ (AZN²)

(c) The standard deviation of $\bar{Y}$ is _____ AZN.

(d) The auditor wants the standard deviation of $\bar{Y}$ to be at most **14** AZN. The smallest sample size that achieves this is $n =$ _____

Problem 5

seed 20260925

A microfinance lender issues $n = 70$ small business loans in one cohort. Each loan defaults with probability $p = 0.14$, independently of the others, so the number of defaults Y is binomial. The observed default rate is $\hat{p} = Y/n$, and each default costs the lender **1500** AZN, so the cohort's credit loss is $L = 1500 Y$ AZN.

Give each answer to at least four significant figures.

(a) $V(\hat{p}) =$ _____

(b) The standard deviation of $\hat{p}$ is _____

(c) $E(L) =$ _____ AZN

(d) The standard deviation of L is _____ AZN.

Problem 6

seed 20260925

A food distributor sells two product lines. In a month it ships Y_1 hundred pallets of line 1 and Y_2 hundred pallets of line 2, and Y_3 hundred pallets come back as returns. Its monthly net cash flow, in thousand AZN, is

$$U = 2Y_1 + 4Y_2 - 3Y_3.$$

The finance team's estimates are

	Y_1	Y_2	Y_3	
mean	38	17	9	$\text{Cov}(Y_1, Y_2) = -10.50, \text{Cov}(Y_1, Y_3) = 1.80, \text{Cov}(Y_2, Y_3) = 1.00.$
variance	36	25	1	

Give each answer to at least four significant figures.

(a) $E(U) =$ _____ thousand AZN

(b) $V(U) =$ _____

(c) The standard deviation of U is _____ thousand AZN.

Problem 7

seed 20260925

A pension fund holds three sleeves: a regional equity fund (Y_1), a corporate bond fund (Y_2) and short-term AZN deposits (Y_3), where Y_i is the monthly return of sleeve i in percent. The weights and the risk model are

	Y_1	Y_2	Y_3	
weight a_i	0.25	0.40	0.35	$\rho_{12} = 0.10, \rho_{13} = -0.25, \rho_{23} = 0.20.$
$E(Y_i)$	1.7	1.1	0.4	
σ_i	9.5	7.0	3.5	

Let $U = a_1Y_1 + a_2Y_2 + a_3Y_3$ be the fund's monthly return. Give each answer to at least four significant figures.

(a) $E(U) =$ _____ (% per month)

(b) $V(U) =$ _____ ($\%^2$)

(c) The standard deviation of U is _____ (% per month)

Problem 8

seed 20260925

A fund manager combines a telecom share (monthly return Y_1 , %) with an electricity-utility share (monthly return Y_2 , %). She puts a fraction a of the money in the telecom share and $1 - a$ in the utility, so the portfolio return is $U = aY_1 + (1 - a)Y_2$. The inputs are

$$E(Y_1) = 2.4, \quad E(Y_2) = 0.6, \quad \sigma_1 = 7.0, \quad \sigma_2 = 6.0, \quad \rho = -0.25.$$

Give each answer to at least four significant figures.

(a) The weight a that makes $V(U)$ as small as possible is $a =$ _____

(b) The smallest achievable variance is $V(U) =$ _____ ($\%^2$)

(c) and the corresponding standard deviation is _____ (% per month)

(d) The expected return of this minimum-variance portfolio is $E(U) =$ _____ (% per month)

Problem 9

seed 20260925

An index fund buys equal amounts of n shares from the same stock market. Each share has a monthly return with standard deviation $\sigma = 7.5\%$, and every pair of shares has correlation $\rho = 0.48$. The fund's return is $U = \frac{1}{n}(Y_1 + \dots + Y_n)$.

Give each answer to at least four significant figures.

- (a) With $n = 6$ shares, $V(U) =$ _____ ($\%^2$)
- (b) With $n = 50$ shares, $V(U) =$ _____ ($\%^2$)
- (c) As $n \rightarrow \infty$, $V(U)$ tends to _____ ($\%^2$)
- (d) so no amount of diversification in this market takes the fund's monthly standard deviation below _____ %.

Problem 10

seed 20260925

An oil-service company in Baku sees its next-month revenue change by Y_1 thousand AZN, with mean 0 and standard deviation $\sigma_1 = 280$. When the oil price falls, its revenue falls. Its treasurer can buy contracts of an inverse oil-price fund: the value of one contract changes by Y_2 thousand AZN over the month, with standard deviation $\sigma_2 = 38$, and Y_1, Y_2 have correlation $\rho = -0.70$. Holding h contracts, the company's total change is $U = Y_1 + hY_2$.

Give each answer to at least four significant figures.

- (a) $\text{Cov}(Y_1, Y_2) =$ _____
- (b) The number of contracts that minimizes $V(U)$ is $h^* =$ _____
- (c) With h^* contracts the standard deviation of U is _____ thousand AZN.
- (d) The fraction of the unhedged variance $V(Y_1)$ that the hedge removes is _____

Problem 11

seed 20260925

Two pension funds run by the same asset manager invest in the same three asset classes: local equities (Y_1), corporate bonds (Y_2) and government bills (Y_3), where Y_i is the monthly return in percent. The covariance matrix $\text{Cov}(Y_i, Y_j)$ is

	Y_1	Y_2	Y_3
Y_1	49	17.50	-4.20
Y_2	17.50	25	0.00
Y_3	-4.20	0.00	4

The growth fund's return is $U_1 = 0.4Y_1 + 0.2Y_2 + 0.4Y_3$ and the conservative fund's is $U_2 = 0.1Y_1 + 0.2Y_2 + 0.7Y_3$.

Give each answer to at least four significant figures.

(a) $\text{Cov}(U_1, U_2) =$ _____

(b) $V(U_1) =$ _____

(c) $V(U_2) =$ _____

(d) The correlation between the two funds' returns is _____

Problem 12

seed 20260925

Next year a commercial bank's average lending yield will be Y_1 and its average cost of funding Y_2 , both in percent a year. Its asset-liability committee estimates

$$E(Y_1) = 14.8, \quad \sigma_1 = 1.6, \quad E(Y_2) = 8.2, \quad \sigma_2 = 1.3, \quad \rho = 0.74.$$

The two rates move together because both follow the central bank's policy rate. The net interest margin is the spread $D = Y_1 - Y_2$.

Give each answer to at least four significant figures.

(a) $V(D) =$ _____

(b) If a junior analyst ignored the correlation and treated Y_1, Y_2 as independent, the figure reported for $V(D)$ would be _____

(c) By Tchebysheff's theorem (Theorem 4.13), the margin lies within two standard deviations of its mean with probability at least 0.75. Using the correct variance from (a), that interval is from _____ to _____ percentage points.

Problem 13

seed 20260925

A mobile operator replaces two components in a rural base station together when either fails. Let Y_1 be the operating life of the power unit and Y_2 that of the microwave radio link, both in years. Their joint density is

$$f(y_1, y_2) = \begin{cases} \frac{1}{18} y_1 e^{-y_1/3 - y_2/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Parts wear and maintenance accrue with operating life, so the cost of the replacement visit, in hundred AZN, is $C = 110 + 5Y_1 + 9Y_2$.

Give each answer to at least four significant figures.

(a) $E(C) =$ _____ hundred AZN

(b) $V(C) =$ _____

(c) The standard deviation of C is _____ hundred AZN.

Problem 14

seed 20260925

A fruit-and-vegetable trader at a Baku bazaar models her daily gain from sales X as normal with mean **480** AZN and standard deviation **80** AZN (X can be low when perishables have to be thrown away). Her daily overhead Y (stall rent share, transport, spoilage handling) has a gamma distribution with $\alpha = 3$ and $\beta = 30$ AZN. X and Y are independent. Her net daily gain is $U = X - Y$.

Give each answer to at least four significant figures.

(a) $E(U) =$ _____ AZN

(b) $V(U) =$ _____

(c) A net gain of **680** AZN or more lies k standard deviations above the mean, where $k =$ _____

(d) Using only the mean and variance, Tchebysheff's theorem gives the upper bound $P(U \geq 680) \leq$ _____

Problem 15

seed 20260925

A central-bank inspector visits a branch that holds $N = 75$ loan files, of which $r = 12$ are misclassified (a non-performing loan recorded as performing). She pulls $n = 9$ files one after another, at random and without replacement. Let $X_k = 1$ if the k th file she pulls is misclassified and $X_k = 0$ otherwise, and let $Y = X_1 + \dots + X_n$.

Give each answer to at least four significant figures.

(a) $P(X_k = 1) =$ _____ (the same for every k)

(b) For $j \neq k$, $\text{Cov}(X_j, X_k) =$ _____

(c) $E(Y) =$ _____

(d) $V(Y) =$ _____

Problem 16

seed 20260925

In a bank's corporate department, Y_1 and Y_2 are the fractions of a working day that relationship managers I and II spend with clients. Their joint density is

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The department's daily bonus pool, in AZN, is $W = 180 + 28Y_1 + 58Y_2$.

Give each answer to at least four significant figures.

(a) $\text{Cov}(28Y_1, 58Y_2) =$ _____

(b) $E(W) =$ _____ AZN

(c) $V(W) =$ _____

(d) By Tchebysheff's theorem the pool lies within two standard deviations of its mean on at least 75% of days, that is between _____ and _____ AZN.

Problem 17

seed 20260925

A robo-adviser offered by a Baku bank builds each client portfolio from an equity ETF (monthly return Y_1 , %) and a sukuk fund (monthly return Y_2 , %), with a fraction a in the ETF and $1 - a$ in the sukuk fund, no short positions. Its inputs are

$$E(Y_1) = 1.6, \quad \sigma_1 = 11.0, \quad E(Y_2) = 0.55, \quad \sigma_2 = 3.25, \quad \rho = -0.05.$$

A client asks for an expected return of **1.39%** per month.

Give each answer to at least four significant figures.

(a) The weight that delivers exactly this expected return is $a =$ _____

(b) The standard deviation of that portfolio's return is _____ (% per month)

(c) The weight of the minimum-variance portfolio is $a =$ _____

(d) Its standard deviation is _____ (% per month)

(e) and its expected return is _____ (% per month).

Problem 18

seed 20260925

An equity fund puts equal amounts into each of n shares listed on regional exchanges. The shares differ in risk, but across the fund's universe the analyst has estimated two averages of their monthly returns (in %):

- the average of the n variances, $\bar{v} = \frac{1}{n} \sum_i V(Y_i) = 75$,
- the average of the $n(n-1)/2$ pairwise covariances, $\bar{c} = \frac{1}{\binom{n}{2}} \sum_{i < j} \text{Cov}(Y_i, Y_j)$

and she treats both averages as the same whatever n is. The fund's return is $U = \frac{1}{n} \sum_{i=1}^n Y_i$.

Give each answer to at least four significant figures.

(a) The fund now holds $n = 9$ shares. $V(U) =$ _____

(b) Its monthly standard deviation is _____ %.

(c) As $n \rightarrow \infty$, $V(U)$ tends to _____

(d) The smallest n for which $V(U)$ is no more than **25%** above that limit is $n =$ _____

Problem 19

seed 20260925

An analyst studies $n = 31$ months of returns $Y_1, \dots, Y_n$ (in %) of an AZN money-market fund, modelled as independent with common mean $\mu = 0.7$ and standard deviation $\sigma = 7.6$. Let $\bar{Y}$ be the sample mean. She wants to know how the first month's return relates to the average it is part of.

Give each answer to at least four significant figures.

(a) $\text{Cov}(\bar{Y}, Y_1) =$ _____

(b) $V(Y_1 - \bar{Y}) =$ _____

(c) $V(Y_1 + \bar{Y}) =$ _____

(d) The correlation between Y_1 and $\bar{Y}$ is _____

Problem 20

seed 20260925

An investment company holds a long portfolio $U_1 = 0.3Y_1 + 0.7Y_2$, where Y_1 and Y_2 are the monthly returns (in %) of a bank-share index and an energy-share index with

$$\sigma_1 = 10.0, \quad \sigma_2 = 4.5, \quad \rho = 0.75.$$

Its trading desk runs a separate long-short position $U_2 = Y_1 - hY_2$: long the bank index, short h units of the energy index.

Give each answer to at least four significant figures.

(a) With $h = 1$, $\text{Cov}(U_1, U_2) =$ _____

(b) The value of h that makes U_2 uncorrelated with U_1 is $h =$ _____

(c) With that h , $V(U_2) =$ _____

(d) and the variance of the combined return is $V(U_1 + U_2) =$ _____
