Week 14, Problem Set 1 - worked solutions

Wackerly 5.8 - The Expected Value and Variance of Linear Functions; Portfolio Variance

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

An investor in Baku splits her savings between shares of a listed bank and an AZN government-bond fund, with a fraction $a_1 = 0.30$ in the bank share and $a_2 = 0.70$ in the bond fund. Let Y_1 and Y_2 be the two monthly returns, in percent. Her analyst estimates

$$E(Y_1) = 2.2, \quad E(Y_2) = 0.6, \quad \sigma_1 = 11.5, \quad \sigma_2 = 4.0, \quad \rho = 0.60.$$

The portfolio return is $U = a_1Y_1 + a_2Y_2$. Give each answer to at least four significant figures.

(a) $E(U) =$ ______________ (% per month)

(b) $V(U) =$ ______________ ($\%^2$)

(c) The standard deviation of U is ______________ (% per month)

(a) Theorem 5.12(a): $E(U) = a_1E(Y_1) + a_2E(Y_2) = 0.30(2.2) + 0.70(0.6) = 1.0800$.

(b) Theorem 5.12(b) needs the covariance, which comes from the correlation: $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = 0.60(11.5)(4.0) = 27.6000$. Then

$$V(U) = a_1^2V(Y_1) + a_2^2V(Y_2) + 2a_1a_2\text{Cov}(Y_1, Y_2)$$
$$= 0.30^2(11.5)^2 + 0.70^2(4.0)^2 + 2(0.30)(0.70)(27.6000) = 11.9025 + 7.8400 + (11.5920) = 31.3345.$$

(c) $\sqrt{31.3345} = 5.5977$ percent per month.

The mean is a weighted average of the two means, but the standard deviation is not a weighted average of σ_1 and σ_2: it is at most that average, with equality only when $\rho = 1$. The gap is the diversification benefit.

Problem 2

seed 20260925

A construction contractor in Sumgait buys cement at 110 AZN per tonne and steel rebar at 1450 AZN per tonne. Let Y_1 and Y_2 be the tonnes of cement and of rebar bought in a week, with

$$E(Y_1) = 42, \quad V(Y_1) = 60, \qquad E(Y_2) = 8, \quad V(Y_2) = 4.0.$$

The weekly spending is $C = 110Y_1 + 1450Y_2$ AZN. Give each answer to at least four significant figures.

(a) $E(C) =$ ______________ AZN

A junior analyst treats Y_1 and Y_2 as independent.

(b) Under that assumption, $V(C) =$ ______________ (AZN^2)

(c) and the standard deviation of C is ______________ AZN.

(d) Both purchases rise and fall with the number of building sites that are active that week, and the records show a correlation $\rho = 0.6$ between Y_1 and Y_2. With this correlation, $V(C) =$ ______________ (AZN^2)

(a) Theorem 5.12(a) with $a_1 = 110$, $a_2 = 1450$: $E(C) = 110(42) + 1450(8) = 16220$ AZN. The mean needs no assumption about dependence at all.

(b) Theorem 5.12(b) with $\text{Cov}(Y_1, Y_2) = 0$:

$$V(C) = 110^2(60) + 1450^2(4.0) = 726000 + 8410000 = 9136000.$$

Notice the prices enter squared.

(c) $\sqrt{9136000} = 3022.58$ AZN.

(d) Now $\text{Cov}(Y_1, Y_2) = \rho\sqrt{V(Y_1)V(Y_2)} = 0.6\sqrt{60 \times 4.0} = 9.29516$, and the cross term of Theorem 5.12(b) is $2(110)(1450)(9.29516) = 2965156.0$. So

$$V(C) = 9136000 + 2965156.0 = 12101156.0.$$

A shared driver makes the two purchases move together, which is exactly what a positive covariance term adds to the risk of the total. Assuming independence understated the spending variance.

Problem 3

seed 20260925

A restaurant on Nizami Street records its daily revenue Y_1 and its daily food cost Y_2, both in AZN. The owner's figures are

$$E(Y_1) = 2600, \quad V(Y_1) = 250000, \qquad E(Y_2) = 1100, \quad V(Y_2) = 44100, \qquad \text{Cov}(Y_1, Y_2) = 57750.$$

Rent and wages add a fixed 600 AZN a day, so the daily operating profit is $U = Y_1 - Y_2 - 600$. Give each answer to at least four significant figures.

(a) $E(U) =$ ____________ AZN

(b) $V(U) =$ ____________ (AZN^2)

(c) The standard deviation of the daily profit is ____________ AZN.

(d) For comparison, the variance of the total cash flowing through the till and the kitchen, $V(Y_1 + Y_2) =$ ____________

Write $U = a_1Y_1 + a_2Y_2 + c$ with $a_1 = 1$, $a_2 = -1$ and the constant $c = -600$. A constant shifts the mean and has no effect on the variance.

(a) $E(U) = 2600 - 1100 - 600 = 900$ AZN.

(b) Theorem 5.12(b):

$$V(U) = (1)^2V(Y_1) + (-1)^2V(Y_2) + 2(1)(-1)\text{Cov}(Y_1, Y_2) = 250000 + 44100 - 2(57750) = 178600.$$

(c) $\sqrt{178600} = 422.611$ AZN.

(d) With $a_2 = +1$ the cross term changes sign: $V(Y_1 + Y_2) = 250000 + 44100 + 2(57750) = 409600$.

Busy days bring both more revenue and more food cost, so the covariance is positive. That co-movement makes the **difference** steadier than either component alone would suggest and the **sum** more volatile.

Problem 4

seed 20260925

The amount of a single cash withdrawal at a bank's ATMs in Ganja has mean $\mu = 100$ AZN and standard deviation $\sigma = 115$ AZN. An auditor draws $n = 55$ withdrawals at random from the log and treats their amounts $Y_1, \ldots, Y_n$ as independent. Let $\overline{Y}$ be the sample mean. Give each answer to at least four significant figures.

(a) $E(\overline{Y}) =$ ____________ AZN

(b) $V(\overline{Y}) =$ ____________ (AZN^2)

(c) The standard deviation of $\overline{Y}$ is ____________ AZN.

(d) The auditor wants the standard deviation of $\overline{Y}$ to be at most 14 AZN. The smallest sample size that achieves this is $n =$ ____________

$\overline{Y} = \frac{1}{n}Y_1 + \cdots + \frac{1}{n}Y_n$ is a linear function with every $a_i = 1/n$ (Example 5.27).

(a) Theorem 5.12(a): $E(\overline{Y}) = \sum_{i=1}^{n} \frac{1}{n}\mu = \mu = 100$ AZN.

(b) Theorem 5.12(b). By independence every covariance is zero, so

$$V(\overline{Y}) = \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n} = \frac{13225}{55} = 240.4545.$$

(c) $\sigma/\sqrt{n} = 115/\sqrt{55} = 15.5066$ AZN.

(d) We need $\sigma/\sqrt{n} \le 14$, i.e. $n \ge \sigma^2/14^2 = 13225/196 = 67.4745$. The smallest whole number that satisfies this is $n = 68$.

The standard deviation of an average falls only like $1/\sqrt{n}$: halving it costs four times as many withdrawals.

Problem 5

seed 20260925

A microfinance lender issues $n = 70$ small business loans in one cohort. Each loan defaults with probability $p = 0.14$, independently of the others, so the number of defaults Y is binomial. The observed default rate is $\hat{p} = Y/n$, and each default costs the lender 1500 AZN, so the cohort's credit loss is $L = 1500\,Y$ AZN.

Give each answer to at least four significant figures.

(a) $V(\hat{p}) =$ ______________

(b) The standard deviation of $\hat{p}$ is ______________

(c) $E(L) =$ ______________ AZN

(d) The standard deviation of L is ______________ AZN.

For a binomial count, $E(Y) = np = 70(0.14) = 9.80$ and $V(Y) = npq = 70(0.14)(0.86) = 8.4280$. Both $\hat{p}$ and L are linear functions $a_1 Y$ of the single variable Y, so Theorem 5.12 gives $E(a_1 Y) = a_1 E(Y)$ and $V(a_1 Y) = a_1^2 V(Y)$ (Example 5.28).

(a) With $a_1 = 1/n$: $V(\hat{p}) = \frac{1}{n^2} npq = \frac{pq}{n} = \frac{0.14(0.86)}{70} = 0.00172000$. (And $E(\hat{p}) = p$: the observed rate is unbiased for the true one.)

(b) $\sqrt{0.00172000} = 0.041473$.

(c) With $a_1 = 1500$: $E(L) = 1500(9.80) = 14700.00$ AZN.

(d) $V(L) = 1500^2(8.4280)$, so the standard deviation is $1500\sqrt{8.4280} = 4354.653$ AZN. Scaling a variable by a scales its standard deviation by $|a|$.

Problem 6

seed 20260925

A food distributor sells two product lines. In a month it ships Y_1 hundred pallets of line 1 and Y_2 hundred pallets of line 2, and Y_3 hundred pallets come back as returns. Its monthly net cash flow, in thousand AZN, is

$$U = 2Y_1 + 4Y_2 - 3Y_3.$$

The finance team's estimates are

	Y_1	Y_2	Y_3
mean	38	17	9
variance	36	25	1

$$\mathrm{Cov}(Y_1, Y_2) = -10.50, \quad \mathrm{Cov}(Y_1, Y_3) = 1.80, \quad \mathrm{Cov}(Y_2, Y_3) = 1.00.$$

Give each answer to at least four significant figures.

(a) $E(U) =$ ______________ thousand AZN

(b) $V(U) =$ ______________

(c) The standard deviation of U is ______________ thousand AZN.

The coefficients are $a_1 = 2$, $a_2 = 4$, $a_3 = -3$. Keep the minus sign on a_3: it flips the sign of every cross term involving Y_3.

(a) Theorem 5.12(a): $E(U) = 2(38) + 4(17) - 3(9) = 117$.

(b) Theorem 5.12(b), with the three pairs $(1,2)$, $(1,3)$, $(2,3)$:

$$V(U) = a_1^2 V(Y_1) + a_2^2 V(Y_2) + a_3^2 V(Y_3) + 2a_1a_2\mathrm{Cov}(Y_1, Y_2) + 2a_1a_3\mathrm{Cov}(Y_1, Y_3) + 2a_2a_3\mathrm{Cov}(Y_2, Y_3).$$

The variance terms are $2^2(36) = 144$, $4^2(25) = 400$, $3^2(1) = 9$. The cross terms are

$$2(2)(4)(-10.50) = -168.00, \quad 2(2)(-3)(1.80) = -21.60, \quad 2(4)(-3)(1.00) = -24.00.$$

Adding all six: $V(U) = 339.40$.

(c) $\sqrt{339.40} = 18.4228$ thousand AZN.

Problem 7

seed 20260925

A pension fund holds three sleeves: a regional equity fund (Y_1), a corporate bond fund (Y_2) and short-term AZN deposits (Y_3), where Y_i is the monthly return of sleeve i in percent. The weights and the risk model are

	Y_1	Y_2	Y_3
weight a_i	0.25	0.40	0.35
$E(Y_i)$	1.7	1.1	0.4
σ_i	9.5	7.0	3.5

$$\rho_{12} = 0.10, \quad \rho_{13} = -0.25, \quad \rho_{23} = 0.20.$$

Let $U = a_1Y_1 + a_2Y_2 + a_3Y_3$ be the fund's monthly return. Give each answer to at least four significant figures.

(a) $E(U) =$ ______________ (% per month)

(b) $V(U) =$ ______________ ($\%^2$)

(c) The standard deviation of U is ______________ (% per month)

(a) Theorem 5.12(a): $E(U) = 0.25(1.7) + 0.40(1.1) + 0.35(0.4) = 1.0050$.

(b) First the covariances, $\text{Cov}(Y_i, Y_j) = \rho_{ij}\sigma_i\sigma_j$:

$$\text{Cov}(Y_1, Y_2) = 6.6500, \quad \text{Cov}(Y_1, Y_3) = -8.3125, \quad \text{Cov}(Y_2, Y_3) = 4.9000.$$

Theorem 5.12(b) has three variance terms and one cross term for each of the three pairs \(i
(c) $\sqrt{16.2286} = 4.0285$ percent per month.

With three assets there are three covariance terms; with n assets there are $n(n-1)/2$ of them, which is why covariances rather than variances come to dominate the risk of a large portfolio.

Problem 8

seed 20260925

A fund manager combines a telecom share (monthly return Y_1, %) with an electricity-utility share (monthly return Y_2, %). She puts a fraction a of the money in the telecom share and $1-a$ in the utility, so the portfolio return is $U = aY_1 + (1-a)Y_2$. The inputs are

$$E(Y_1) = 2.4, \quad E(Y_2) = 0.6, \quad \sigma_1 = 7.0, \quad \sigma_2 = 6.0, \quad \rho = -0.25.$$

Give each answer to at least four significant figures.

(a) The weight a that makes $V(U)$ as small as possible is $a =$ ______________

(b) The smallest achievable variance is $V(U) =$ ______________ ($\%^2$)

(c) and the corresponding standard deviation is ______________ (% per month)

(d) The expected return of this minimum-variance portfolio is $E(U) =$ ______________ (% per month)

Let $c = \text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = -0.25(7.0)(6.0) = -10.5000$. By Theorem 5.12(b),

$$V(a) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2 + 2a(1-a)c.$$

(a) This is a quadratic in a with positive leading coefficient $\sigma_1^2 + \sigma_2^2 - 2c = V(Y_1 - Y_2) > 0$. Setting the derivative to zero,

$$V'(a) = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 + 2(1-2a)c = 0 \quad \Longrightarrow \quad a^* = \frac{\sigma_2^2 - c}{\sigma_1^2 + \sigma_2^2 - 2c} = \frac{36.00 - (-10.5000)}{49.00 + 36.00 - 2(-10.5000)} = \frac{46.5000}{106.0000} = 0.43868.$$

(b) Substituting $a^* = 0.43868$ and $1 - a^* = 0.56132$ into $V(a)$ gives $V(a^*) = 15.6014$. A shortcut that follows from the algebra is

$$V(a^*) = \frac{\sigma_1^2\sigma_2^2 - c^2}{\sigma_1^2 + \sigma_2^2 - 2c} = \frac{1764.0000 - 110.2500}{106.0000} = 15.6014.$$

(c) $\sqrt{15.6014} = 3.9499$.

(d) Theorem 5.12(a): $E(U) = 0.43868(2.4) + 0.56132(0.6) = 1.3896$.

The minimum-variance portfolio is less volatile than the safer asset alone ($\sigma_2 = 6.0$) because the correlation is below σ_2/σ_1: adding a little of the risky share lowers risk.

Problem 9

seed 20260925

An index fund buys equal amounts of n shares from the same stock market. Each share has a monthly return with standard deviation $\sigma = 7.5\%$, and every pair of shares has correlation $\rho = 0.48$. The fund's return is $U = \frac{1}{n}(Y_1 + \cdots + Y_n)$.

Give each answer to at least four significant figures.

(a) With $n = 6$ shares, $V(U) =$ ______________ ($\%^2$)

(b) With $n = 50$ shares, $V(U) =$ ______________ ($\%^2$)

(c) As $n \to \infty$, $V(U)$ tends to ______________ ($\%^2$)

(d) so no amount of diversification in this market takes the fund's monthly standard deviation below ______________ %.

Every $a_i = 1/n$, every $V(Y_i) = \sigma^2 = 56.25$ and every $\text{Cov}(Y_i, Y_j) = \rho\sigma^2 = 0.48(56.25) = 27.0000$. In Theorem 5.12(b) there are n variance terms and $n(n-1)/2$ pairs \(i
(a) For $n = 6$ (there are 15 pairs): $V(U) = 56.25/6 + (1 - 1/6)(27.0000) = 9.3750 + 22.5000 = 31.8750$.

(b) For $n = 50$: $V(U) = 27.5850$.

(c) The first term vanishes and the second tends to the common covariance: $\rho\sigma^2 = 27.0000$.

(d) $\sqrt{27.0000} = 5.1962\%$.

The own-variance part of the risk is diversified away at rate $1/n$; the covariance part is market risk that every share carries and that averaging cannot remove. When $\rho = 0$ (Example 5.27) the limit would be zero.

Problem 10

seed 20260925

An oil-service company in Baku sees its next-month revenue change by Y_1 thousand AZN, with mean 0 and standard deviation $\sigma_1 = 280$. When the oil price falls, its revenue falls. Its treasurer can buy contracts of an inverse oil-price fund: the value of one contract changes by Y_2 thousand AZN over the month, with standard deviation $\sigma_2 = 38$, and Y_1, Y_2 have correlation $\rho = -0.70$. Holding h contracts, the company's total change is $U = Y_1 + hY_2$.

Give each answer to at least four significant figures.

(a) $\text{Cov}(Y_1, Y_2) =$ ______________

(b) The number of contracts that minimizes $V(U)$ is $h^* =$ ______________

(c) With h^* contracts the standard deviation of U is ______________ thousand AZN.

(d) The fraction of the unhedged variance $V(Y_1)$ that the hedge removes is ______________

(a) $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = (-0.70)(280)(38) = -7448.00$.

(b) Theorem 5.12(b) with $a_1 = 1$, $a_2 = h$:

$$V(U) = \sigma_1^2 + h^2\sigma_2^2 + 2h\,\text{Cov}(Y_1, Y_2).$$

Setting $dV/dh = 2h\sigma_2^2 + 2\text{Cov}(Y_1, Y_2) = 0$,

$$h^* = -\frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_2^2} = -\frac{-7448.00}{1444} = 5.1579,$$

and the second derivative $2\sigma_2^2 > 0$ confirms a minimum. The negative covariance makes h^* positive: the company **buys** the inverse fund.

(c) Substituting $h^* = -\rho\sigma_1/\sigma_2$,

$$V(U) = \sigma_1^2 + \rho^2\sigma_1^2 - 2\rho^2\sigma_1^2 = \sigma_1^2(1-\rho^2) = 78400(1-(-0.70)^2) = 39984.00,$$

so the standard deviation is $\sqrt{39984.00} = 199.960$ thousand AZN.

(d) The hedged variance is $(1-\rho^2)$ times the unhedged one, so the fraction removed is $\rho^2 = 0.4900$. How much risk a hedge can take out depends only on how tightly the instrument tracks the exposure.

Problem 11

seed 20260925

Two pension funds run by the same asset manager invest in the same three asset classes: local equities (Y_1), corporate bonds (Y_2) and government bills (Y_3), where Y_i is the monthly return in percent. The covariance matrix $\mathrm{Cov}(Y_i, Y_j)$ is

	Y_1	Y_2	Y_3
Y_1	49	17.50	−4.20
Y_2	17.50	25	0.00
Y_3	−4.20	0.00	4

The growth fund's return is $U_1 = 0.4Y_1 + 0.2Y_2 + 0.4Y_3$ and the conservative fund's is $U_2 = 0.1Y_1 + 0.2Y_2 + 0.7Y_3$.

Give each answer to at least four significant figures.

(a) $\mathrm{Cov}(U_1, U_2) =$ ______________

(b) $V(U_1) =$ ______________

(c) $V(U_2) =$ ______________

(d) The correlation between the two funds' returns is ______________

(a) Theorem 5.12(c): $\mathrm{Cov}(U_1, U_2) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \mathrm{Cov}(Y_i, Y_j)$, nine terms, with $\mathrm{Cov}(Y_i, Y_i) = V(Y_i)$. It is tidiest to multiply each row of the matrix by the b weights first:

$$\begin{aligned} g_1 &= 0.1(49) + 0.2(17.50) + 0.7(-4.20) = 5.4600, \\ g_2 &= 0.1(17.50) + 0.2(25) + 0.7(0.00) = 6.7500, \\ g_3 &= 0.1(-4.20) + 0.2(0.00) + 0.7(4) = 2.3800, \end{aligned}$$

then weight by a: $\mathrm{Cov}(U_1, U_2) = 0.4(5.4600) + 0.2(6.7500) + 0.4(2.3800) = 4.4860$.

(b) Theorem 5.12(b) with the a weights (the special case $b = a$ of part (c)):

$$V(U_1) = 0.4^2(49) + 0.2^2(25) + 0.4^2(4) + 2(0.4)(0.2)(17.50) + 2(0.4)(0.4)(-4.20) + 2(0.2)(0.4)(0.00) = 10.9360.$$

(c) The same with the b weights: $V(U_2) = 3.5620$.

(d)

$$\rho = \frac{\mathrm{Cov}(U_1, U_2)}{\sqrt{V(U_1)V(U_2)}} = \frac{4.4860}{\sqrt{10.9360 \times 3.5620}} = 0.7188.$$

Two funds with different labels can still be closely correlated when they hold the same assets, so offering both does less for a saver's diversification than the names suggest.

Problem 12

seed 20260925

Next year a commercial bank's average lending yield will be Y_1 and its average cost of funding Y_2, both in percent a year. Its asset-liability committee estimates

$$E(Y_1) = 14.8, \quad \sigma_1 = 1.6, \qquad E(Y_2) = 8.2, \quad \sigma_2 = 1.3, \qquad \rho = 0.74.$$

The two rates move together because both follow the central bank's policy rate. The net interest margin is the spread $D = Y_1 - Y_2$.

Give each answer to at least four significant figures.

(a) $V(D) =$ ____________

(b) If a junior analyst ignored the correlation and treated Y_1, Y_2 as independent, the figure reported for $V(D)$ would be ____________

(c) By Tchebysheff's theorem (Theorem 4.13), the margin lies within two standard deviations of its mean with probability at least 0.75. Using the correct variance from (a), that interval is from ____________ to ____________ percentage points.

$E(D) = 14.8 - 8.2 = 6.6$ by Theorem 5.12(a), and $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = 0.74(1.6)(1.3) = 1.5392$.

(a) Theorem 5.12(b) with $a_1 = 1$, $a_2 = -1$ (Example 5.26):

$$V(D) = V(Y_1) + V(Y_2) - 2\text{Cov}(Y_1, Y_2) = 2.56 + 1.69 - 2(1.5392) = 1.1716.$$

(b) Dropping the covariance term, $2.56 + 1.69 = 4.2500$. Because the rates are positively correlated, ignoring the correlation overstates the margin's risk: the part of each rate that comes from the policy rate cancels in the spread.

(c) $\sigma_D = \sqrt{1.1716} = 1.0824$, so the interval $E(D) \pm 2\sigma_D$ is $6.6 \pm 2(1.0824)$, that is from 4.4352 to 8.7648. Tchebysheff needs no distributional assumption, only the mean and variance that Theorem 5.12 supplies.

Problem 13

seed 20260925

A mobile operator replaces two components in a rural base station together when either fails. Let Y_1 be the operating life of the power unit and Y_2 that of the microwave radio link, both in years. Their joint density is

$$f(y_1, y_2) = \begin{cases} \frac{1}{18}\, y_1\, e^{-y_1/3 - y_2/2}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Parts wear and maintenance accrue with operating life, so the cost of the replacement visit, in hundred AZN, is $C = 110 + 5Y_1 + 9Y_2$.

Give each answer to at least four significant figures.

(a) $E(C) =$ ____________ hundred AZN

(b) $V(C) =$ ____________

(c) The standard deviation of C is ____________ hundred AZN.

The density factors as

$$f(y_1, y_2) = \left[\frac{y_1\, e^{-y_1/3}}{\Gamma(2)\, 3^2}\right] \left[frac12 e^{-y_2/2}\right], \qquad \Gamma(2)\, 3^2\, 2 = 18,$$

on a rectangular (product) support, so by Theorem 5.5 Y_1 and Y_2 are independent, with Y_1 gamma ($\alpha = 2$, $\beta = 3$) and Y_2 exponential with $\beta = 2$. Hence $E(Y_1) = \alpha\beta = 6$, $V(Y_1) = \alpha\beta^2 = 18$, $E(Y_2) = 2$, $V(Y_2) = 4$, and $\text{Cov}(Y_1, Y_2) = 0$.

(a) Theorem 5.12(a), the constant carried through: $E(C) = 110 + 5(6) + 9(2) = 158$.

(b) Theorem 5.12(b); the constant drops out and the cross term is zero: $V(C) = 5^2(18) + 9^2(4) = 450 + 324 = 774$.

(c) $\sqrt{774} = 27.8209$ hundred AZN.

Problem 14

seed 20260925

A fruit-and-vegetable trader at a Baku bazaar models her daily gain from sales X as normal with mean 480 AZN and standard deviation 80 AZN (X can be low when perishables have to be thrown away). Her daily overhead Y (stall rent share, transport, spoilage handling) has a gamma distribution with $\alpha = 3$ and $\beta = 30$ AZN. X and Y are independent. Her net daily gain is $U = X - Y$.

Give each answer to at least four significant figures.

(a) $E(U) =$ ____________ AZN

(b) $V(U) =$ ____________

(c) A net gain of 680 AZN or more lies k standard deviations above the mean, where $k =$ ____________

(d) Using only the mean and variance, Tchebysheff's theorem gives the upper bound $P(U \geq 680) \leq$ ____________

For the gamma overhead, $E(Y) = \alpha\beta = 3(30) = 90$ and $V(Y) = \alpha\beta^2 = 3(30)^2 = 2700$.

(a) Theorem 5.12(a) with $a_1 = 1$, $a_2 = -1$: $E(U) = 480 - 90 = 390$ AZN.

(b) Theorem 5.12(b); independence makes the covariance zero, and $(-1)^2 = 1$, so the variances **add** even though the variables are subtracted: $V(U) = 6400 + 2700 = 9100$.

(c) $\sigma_U = \sqrt{9100} = 95.3939$, so $k = (680 - 390)/95.3939 = 290/95.3939 = 3.0400$.

(d) $\{U \geq 680\} \subset \{|U - \mu_U| \geq k\sigma_U\}$, and Theorem 4.13 bounds the probability of the larger event by $1/k^2$:

$$P(U \geq 680) \leq \frac{1}{k^2} = \frac{1}{3.0400^2} = 0.1082.$$

The bound is crude (it ignores that U is nearly symmetric and counts both tails), but it needs nothing beyond what Theorem 5.12 delivered.

Problem 15

seed 20260925

A central-bank inspector visits a branch that holds $N = 75$ loan files, of which $r = 12$ are misclassified (a non-performing loan recorded as performing). She pulls $n = 9$ files one after another, at random and without replacement. Let $X_k = 1$ if the kth file she pulls is misclassified and $X_k = 0$ otherwise, and let $Y = X_1 + \cdots + X_n$.

Give each answer to at least four significant figures.

(a) $P(X_k = 1) =$ ____________ (the same for every k)

(b) For $j \neq k$, $\mathrm{Cov}(X_j, X_k) =$ ____________

(c) $E(Y) =$ ____________

(d) $V(Y) =$ ____________

(a) By symmetry (Example 5.29) every draw is equally likely to be any of the N files, so $P(X_k = 1) = r/N = 12/75 = 0.160000$.

(b) $X_jX_k = 1$ exactly when both files are misclassified, so $E(X_jX_k) = P(X_j = 1, X_k = 1) = \dfrac{r(r-1)}{N(N-1)} = \dfrac{12(11)}{75(74)} = 0.023784$. Then

$$\mathrm{Cov}(X_j, X_k) = E(X_jX_k) - E(X_j)E(X_k) = 0.023784 - 0.160000^2 = -0.00181622.$$

It is negative: finding a misclassified file leaves fewer for the later draws.

(c) Theorem 5.12(a) with every $a_i = 1$: $E(Y) = n\,r/N = 9(0.160000) = 1.44000$.

(d) $V(X_k) = p(1-p) = 0.134400$. Theorem 5.12(b) has n variance terms and $n(n-1)/2 = 36$ equal covariance terms:

$$V(Y) = n\,p(1-p) + 2 \cdot \frac{n(n-1)}{2}\,\mathrm{Cov}(X_j, X_k) = 9(0.134400) + 9(9-1)(-0.00181622) = 1.07883.$$

This agrees with the hypergeometric variance $n\frac{r}{N}\frac{N-r}{N}\frac{N-n}{N-1}$, obtained here without touching the hypergeometric probabilities.

Problem 16

seed 20260925

In a bank's corporate department, Y_1 and Y_2 are the fractions of a working day that relationship managers I and II spend with clients. Their joint density is

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The department's daily bonus pool, in AZN, is $W = 180 + 28Y_1 + 58Y_2$.

Give each answer to at least four significant figures.

(a) $\mathrm{Cov}(28Y_1, 58Y_2) =$ ______________

(b) $E(W) =$ ______________ AZN

(c) $V(W) =$ ______________

(d) By Tchebysheff's theorem the pool lies within two standard deviations of its mean on at least 75% of days, that is between ______________ and ______________ AZN.

(a) The marginal of Y_1 is $f_1(y_1) = \int_0^1 (y_1 + y_2)\,dy_2 = y_1 + \frac{1}{2}$ on $[0, 1]$, so

$$E(Y_1) = \int_0^1 y_1\left(y_1 + \tfrac{1}{2}\right)dy_1 = \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{12}, \qquad E(Y_1^2) = \int_0^1 y_1^2\left(y_1 + \tfrac{1}{2}\right)dy_1 = \tfrac{1}{4} + \tfrac{1}{6} = \tfrac{5}{12},$$

and $V(Y_1) = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$. By symmetry Y_2 has the same mean and variance. Next

$$E(Y_1Y_2) = \int_0^1\int_0^1 y_1y_2(y_1 + y_2)\,dy_1\,dy_2 = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3},$$

so $\mathrm{Cov}(Y_1, Y_2) = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$. By Theorem 5.12(c) with one term on each side, $\mathrm{Cov}(28Y_1, 58Y_2) = 28(58)\left(-\frac{1}{144}\right) = -11.27778$.

(b) Theorem 5.12(a): $E(W) = 180 + (28 + 58)\frac{7}{12} = 180 + 86 \cdot \frac{7}{12} = 230.1667$.

(c) Theorem 5.12(b):

$$V(W) = (28^2 + 58^2)\tfrac{11}{144} + 2(28)(58)\left(-\tfrac{1}{144}\right) = \frac{42380}{144} = 294.3056.$$

(d) $\sigma_W = 17.1553$, and $E(W) \pm 2\sigma_W$ runs from 195.8560 to 264.4773 AZN (Theorem 4.13 with $k = 2$).

Problem 17

seed 20260925

A robo-adviser offered by a Baku bank builds each client portfolio from an equity ETF (monthly return Y_1, %) and a sukuk fund (monthly return Y_2, %), with a fraction a in the ETF and $1 - a$ in the sukuk fund, no short positions. Its inputs are

$$E(Y_1) = 1.6, \quad \sigma_1 = 11.0, \qquad E(Y_2) = 0.55, \quad \sigma_2 = 3.25, \qquad \rho = -0.05.$$

A client asks for an expected return of 1.39% per month.

Give each answer to at least four significant figures.

(a) The weight that delivers exactly this expected return is $a =$ ______________

(b) The standard deviation of that portfolio's return is ______________ (% per month)

(c) The weight of the minimum-variance portfolio is $a =$ ______________

(d) Its standard deviation is ______________ (% per month)

(e) and its expected return is ______________ (% per month).

Write $U(a) = aY_1 + (1 - a)Y_2$ and $c = \mathrm{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = -1.7875$.

(a) Theorem 5.12(a) gives $E(U) = a\mu_1 + (1 - a)\mu_2 = \mu_2 + a(\mu_1 - \mu_2)$, which is linear in a. Setting it equal to the target,

$$a = \frac{1.39 - 0.55}{1.6 - 0.55} = \frac{0.84}{1.05} = 0.80000.$$

(b) Theorem 5.12(b):

$$V(U) = a^2\sigma_1^2 + (1 - a)^2\sigma_2^2 + 2a(1 - a)c = 0.80000^2(121.0000) + 0.20000^2(10.5625) + 2(0.80000)(0.20000)(-1.7875) = 77.2905,$$

so the standard deviation is 8.7915.

(c) Minimising the same quadratic in a ($dV/da = 0$),

$$a^* = \frac{\sigma_2^2 - c}{\sigma_1^2 + \sigma_2^2 - 2c} = \frac{12.3500}{135.1375} = 0.09139.$$

(d) $V(a^*) = 9.4339$, standard deviation 3.0715.

(e) $E(U) = 0.09139(1.6) + 0.90861(0.55) = 0.6460$.

The client's target costs 8.7915 of risk against a floor of 3.0715. Any portfolio with $a < a^*$ would be dominated: it has both a lower mean and a higher variance than the minimum-variance portfolio.

Problem 18

seed 20260925

An equity fund puts equal amounts into each of n shares listed on regional exchanges. The shares differ in risk, but across the fund's universe the analyst has estimated two averages of their monthly returns (in %):

- the average of the n variances, $\bar v = \frac{1}{n}\sum_i V(Y_i) = 75$,
- the average of the $n(n-1)/2$ pairwise covariances, \(\bar c = \binom{n}{2}^{-1}\sum_{i

and she treats both averages as the same whatever n is. The fund's return is $U = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

Give each answer to at least four significant figures.

(a) The fund now holds $n = 9$ shares. $V(U) =$ ____________

(b) Its monthly standard deviation is ____________ %.

(c) As $n \to \infty$, $V(U)$ tends to ____________

(d) The smallest n for which $V(U)$ is no more than 25% above that limit is $n =$ ____________

Theorem 5.12(b) with every $a_i = 1/n$: \[V(U) = \frac{1}{n^2}\sum_{i=1}^n V(Y_i) + \frac{2}{n^2}\sum_{i

(a) $V(U) = 75/9 + (1 - 1/9)(38) = 42.1111$.

(b) $\sqrt{42.1111} = 6.4893$%.

(c) $\bar v/n \to 0$ and $(1 - 1/n)\bar c \to \bar c = 38$. The average covariance is the risk that equal weighting cannot diversify away.

(d) Rewrite $V(U) = \bar c + (\bar v - \bar c)/n$. We need $(\bar v - \bar c)/n \le 0.25\,\bar c$, i.e.

$$n \ge \frac{\bar v - \bar c}{0.25\,\bar c} = \frac{37}{0.25 \times 38} = 3.8947,$$

so the smallest whole n is 4. Check: the bound is $(1 + 0.25)38 = 47.5000$; at $n = 4$, $V(U) = 47.2500$, while at $n = 3$, $V(U) = 50.3333$.

Problem 19

seed 20260925

An analyst studies $n = 31$ months of returns $Y_1, \ldots, Y_n$ (in %) of an AZN money-market fund, modelled as independent with common mean $\mu = 0.7$ and standard deviation $\sigma = 7.6$. Let $\overline{Y}$ be the sample mean. She wants to know how the first month's return relates to the average it is part of.

Give each answer to at least four significant figures.

(a) $\mathrm{Cov}(\overline{Y}, Y_1) =$ ____________

(b) $V(Y_1 - \overline{Y}) =$ ____________

(c) $V(Y_1 + \overline{Y}) =$ ____________

(d) The correlation between Y_1 and $\overline{Y}$ is ____________

Write $\overline{Y} = \sum_j \frac{1}{n} Y_j$. By independence $\mathrm{Cov}(Y_i, Y_j) = 0$ for $i \ne j$ and $\mathrm{Cov}(Y_i, Y_i) = \sigma^2 = 57.76$.

(a) Theorem 5.12(c) with $U_1 = \overline{Y}$ and $U_2 = Y_1$ (one coefficient, $b_1 = 1$): only the $j = 1$ term survives,

$$\mathrm{Cov}(\overline{Y}, Y_1) = \sum_{j=1}^{n} \tfrac{1}{n}\,\mathrm{Cov}(Y_j, Y_1) = \tfrac{1}{n}\sigma^2 = \frac{57.76}{31} = 1.86323.$$

(b) $Y_1 - \overline{Y} = (1 - \frac{1}{n})Y_1 - \frac{1}{n}Y_2 - \cdots - \frac{1}{n}Y_n$, a linear function with independent terms, so by Theorem 5.12(b)

$$V(Y_1 - \overline{Y}) = \left(1 - \tfrac{1}{n}\right)^2\sigma^2 + (n-1)\frac{1}{n^2}\sigma^2 = \frac{(n-1)^2 + (n-1)}{n^2}\sigma^2 = \frac{n-1}{n}\sigma^2 = \frac{30}{31}(57.76) = 55.89677.$$

(Equivalently $V(Y_1) + V(\overline{Y}) - 2\mathrm{Cov}(Y_1, \overline{Y}) = \sigma^2 + \sigma^2/n - 2\sigma^2/n$.)

(c) Now the covariance term does not vanish. By Theorem 5.12(b) applied to the pair $(Y_1, \overline{Y})$,

$$V(Y_1 + \overline{Y}) = V(Y_1) + V(\overline{Y}) + 2\mathrm{Cov}(Y_1, \overline{Y}) = \sigma^2 + \frac{\sigma^2}{n} + \frac{2\sigma^2}{n} = \sigma^2\left(1 + \frac{3}{n}\right) = 63.34968.$$

Contrast (b): by bilinearity (Theorem 5.12(c)), $\mathrm{Cov}(\overline{Y}, Y_1 - \overline{Y}) = \mathrm{Cov}(\overline{Y}, Y_1) - V(\overline{Y}) = \sigma^2/n - \sigma^2/n = 0$. The deviation of an observation from the sample mean is uncorrelated with the sample mean itself, a fact that returns when S^2 is studied in Chapter 7.

(d)

$$\rho = \frac{\sigma^2/n}{\sqrt{\sigma^2 \cdot \sigma^2/n}} = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{31}} = 0.17961.$$

It does not depend on σ or μ at all.

Problem 20

seed 20260925

An investment company holds a long portfolio $U_1 = 0.3Y_1 + 0.7Y_2$, where Y_1 and Y_2 are the monthly returns (in %) of a bank-share index and an energy-share index with

$$\sigma_1 = 10.0, \qquad \sigma_2 = 4.5, \qquad \rho = 0.75.$$

Its trading desk runs a separate long-short position $U_2 = Y_1 - hY_2$: long the bank index, short h units of the energy index.

Give each answer to at least four significant figures.

(a) With $h = 1$, $\mathrm{Cov}(U_1, U_2) =$ ______________

(b) The value of h that makes U_2 uncorrelated with U_1 is $h =$ ______________

(c) With that h, $V(U_2) =$ ______________

(d) and the variance of the combined return is $V(U_1 + U_2) =$ ______________

Let $c = \mathrm{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = 0.75(10.0)(4.5) = 33.7500$. Theorem 5.12(c) with $a = (0.3, 0.7)$ and $b = (1, -h)$ gives four terms:

$$\mathrm{Cov}(U_1, U_2) = 0.3\sigma_1^2 - 0.3h\,c + 0.7c - 0.7h\,\sigma_2^2.$$

(a) With $h = 1$: $0.3(100.00) - 0.3(33.7500) + 0.7(33.7500) - 0.7(20.25) = 29.3250$.

(b) The covariance is linear in h. Setting it to zero,

$$h = \frac{0.3\sigma_1^2 + 0.7c}{0.3c + 0.7\sigma_2^2} = \frac{53.6250}{24.3000} = 2.20679.$$

The denominator is positive because $c > 0$.

(c) Theorem 5.12(b) with $b = (1, -h)$:

$$V(U_2) = \sigma_1^2 + h^2\sigma_2^2 - 2hc = 100.00 + 2.20679^2(20.25) - 2(2.20679)(33.7500) = 49.6576.$$

(d) $V(U_1 + U_2) = V(U_1) + V(U_2) + 2\mathrm{Cov}(U_1, U_2)$, and the last term is now zero. With $V(U_1) = 0.3^2(100.00) + 0.7^2(20.25) + 2(0.3)(0.7)(33.7500) = 33.0975$,

$$V(U_1 + U_2) = 33.0975 + 49.6576 = 82.7551.$$

Making the desk's position uncorrelated with the main portfolio means its risk simply adds, variance on variance, with no cross term to amplify it.

Answer key

Problem	Answers
1	(a) 1.08, (b) 31.3345, (c) 5.59772
2	(a) 16220, (b) 9.136E+06, (c) 3022.58, (d) 1.21012E+07
3	(a) 900, (b) 178600, (c) 422.611, (d) 409600
4	(a) 100, (b) 240.455, (c) 15.5066, (d) 68
5	(a) 0.00172, (b) 0.0414729, (c) 14700, (d) 4354.65
6	(a) 117, (b) 339.4, (c) 18.4228
7	(a) 1.005, (b) 16.2286, (c) 4.02847
8	(a) 0.438679, (b) 15.6014, (c) 3.94986, (d) 1.38962
9	(a) 31.875, (b) 27.585, (c) 27, (d) 5.19615
10	(a) -7448, (b) 5.15789, (c) 199.96, (d) 0.49
11	(a) 4.486, (b) 10.936, (c) 3.562, (d) 0.718759
12	(a) 1.1716, (b) 4.25, (c) 4.43519, (d) 8.76481
13	(a) 158, (b) 774, (c) 27.8209
14	(a) 390, (b) 9100, (c) 3.04003, (d) 0.108205
15	(a) 0.16, (b) -0.00181622, (c) 1.44, (d) 1.07883
16	(a) -11.2778, (b) 230.167, (c) 294.306, (d) 195.856, (e) 264.477
17	(a) 0.8, (b) 8.7915, (c) 0.0913884, (d) 3.07146, (e) 0.645958
18	(a) 42.1111, (b) 6.48931, (c) 38, (d) 4
19	(a) 1.86323, (b) 55.8968, (c) 63.3497, (d) 0.179605
20	(a) 29.325, (b) 2.20679, (c) 49.6576, (d) 82.7551