# Week 15, Problem Set 1

Wackerly 5.10-5.11 - The Bivariate Normal Distribution and Conditional Expectations

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

## Problem 1

seed 20260925

A Baku trading company invoices partners in Russia and Georgia. Let $Y_1$ be next month's percentage change in the ruble per US dollar and $Y_2$ the percentage change in the lari per US dollar. The treasury desk models $(Y_1, Y_2)$ as bivariate normal (Section 5.10) with

$$\mu_1 = -0.5, \quad \sigma_1 = 8, \quad \mu_2 = 0, \quad \sigma_2 = 5.5, \quad \rho = 0.6.$$

**(a)** The marginal distribution of $Y_2$ is normal. Its mean is ______________ and its standard deviation is ______________.

**(b)** $\text{Cov}(Y_1, Y_2) =$ ______________

**(c)** Using Table 4, find the probability that the lari weakens by more than 11.55 percent next month, $P(Y_2 > 11.55)$, to four decimal places.

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## Problem 2

seed 20260925

A Baku bank issues standard ($Y_2 = 0$) and premium ($Y_2 = 1$) credit cards. For a randomly chosen cardholder let $Y_1$ be the number of late payments in the past year (the bank caps the count at 2). The joint probability function $p(y_1, y_2)$ is

| | $y_1 = 0$ | $y_1 = 1$ | $y_1 = 2$ |
|---|---|---|---|
| $y_2 = 0$ (standard) | 0.49 | 0.09 | 0.10 |
| $y_2 = 1$ (premium) | 0.21 | 0.08 | 0.03 |

Give every answer to at least four significant figures.

**(a)** $E(Y_1 \mid Y_2 = 0) =$ ______________

**(b)** $E(Y_1 \mid Y_2 = 1) =$ ______________

**(c)** Use Theorem 5.14 and your answers to (a) and (b) to find $E(Y_1)$: ______________

## Problem 3

seed 20260925

Each morning a bank loads $Y_2$ thousand AZN into a street ATM whose capacity is 54 thousand AZN, and $Y_1$ thousand AZN is withdrawn during the day. The withdrawals cannot exceed what was loaded. The joint density is

$$f(y_1, y_2) = \begin{cases} \frac{2}{2916}, & 0 \le y_1 \le y_2 \le 54, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** Find the conditional expectation of the day's withdrawals given the amount loaded, as a function of $y_2$ (for $0 < y_2 \le 54$):

$E(Y_1 \mid Y_2 = y_2) =$ ______________

(enter an expression in $y_2$, typed as y_2)

**(b)** If 48.6 thousand AZN is loaded, the expected withdrawal is ______________ thousand AZN.

**(c)** Use Theorem 5.14 to find the long-run average daily withdrawal, $E(Y_1) =$ ______________ thousand AZN. Give it to at least four significant figures.

## Problem 4

seed 20260925

A motor insurer in Baku classifies drivers as low-risk or high-risk. For a driver whose annual claim rate is $\lambda$, the number of claims $Y$ in a year has a Poisson distribution with mean $\lambda$. The rate itself depends on the driver:

- with probability 0.6 the driver is low-risk and $\lambda = 0.19$;
- with probability 0.4 the driver is high-risk and $\lambda = 0.5$.

A policy is chosen at random from the book. Give every answer to at least four significant figures.

**(a)** $E(Y) =$ ______________

**(b)** The probability that the policy has no claims in the year, $P(Y = 0) =$ ______________

**(c)** The insurer holds 5500 such policies. The expected total number of claims in the book is ______________

## Problem 5

seed 20260925

An agricultural insurer covers orchards in the Guba district against hail. On a given policy the number of claims $N$ in a season has probability function

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(n)$ | 0.64 | 0.18 | 0.15 | 0.03 |

Each claim, whatever the number of claims, has mean 2300 AZN, and the claim sizes are independent of $N$. Let $T$ be the total paid on the policy in a season ($T = 0$ when $N = 0$).

**(a)** $E(T \mid N = 2) =$ ______________ AZN

**(b)** $E(N) =$ ______________

**(c)** $E(T) =$ ______________ AZN

## Problem 6

seed 20260925

A microfinance lender in Ganja has 17 loans maturing each week. Given the week's default probability $p$, the number $Y$ of those loans that default is binomial with parameters $n = 17$ and $p$ (loans default independently). But $p$ moves with local economic conditions from week to week, and the lender models it as uniformly distributed on the interval $(0.05, 0.12)$.

**(a)** Find $E(Y \mid p)$ as a function of $p$: ______________

(enter an expression in p)

**(b)** $E(p) =$ ______________

**(c)** The expected number of defaults in a week, $E(Y) =$ ______________

## Problem 7

seed 20260925

The Ministry of Economy's trade unit models monthly percentage changes in export volumes. Each model below is a **bivariate normal** distribution.

**Model I** (hazelnuts $X_1$, persimmons $X_2$): $\mu_1 = 2.8$, $\sigma_1 = 6$, $\mu_2 = 1.7$, $\sigma_2 = 5$, $\text{Cov}(X_1, X_2) = 0$.

**Model II** (cotton $W_1$, wine $W_2$): $\mu_1 = 2.2$, $\sigma_1 = 9$, $\mu_2 = 2.3$, $\sigma_2 = 2$, $\text{Cov}(W_1, W_2) = -5.4$.

**(a)** In which model are the two export changes independent?

- ○ Model I only
- ○ Model II only
- ○ Both models
- ○ Neither model

**(b)** For the model you chose in (a), use Table 4 to find the probability that **both** export volumes grow by more than their thresholds next month, the first by more than 9.76 percent and the second by more than 8.9 percent. Give it to four decimal places.

______________

**(c)** For the other model, find the correlation coefficient $\rho$ of the two export changes: ______________

## Problem 8

seed 20260925

Azerbaijan imports most of its milling wheat. Let $Y_1$ be next month's percentage change in the world wheat price and $Y_2$ the percentage change in the Baku bread price index. An analyst at the central bank treats $(Y_1, Y_2)$ as bivariate normal with

$$\mu_1 = 2.8, \quad \sigma_1 = 5, \quad \mu_2 = 1.4, \quad \sigma_2 = 2, \quad \rho = 0.28.$$

The wheat price turns out to have changed by $y_1 = 10.3$ percent.

**(a)** $E(Y_2 \mid Y_1 = 10.3) =$ ______________

**(b)** $V(Y_2 \mid Y_1 = 10.3) =$ ____________

**(c)** Using Table 4, find the conditional probability that bread prices rise by more than 5.1968 percent, $P(Y_2 > 5.1968 \mid Y_1 = 10.3)$, to four decimal places.

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## Problem 9

seed 20260925

A Baku-based airline budgets for fuel. Let $Y_1$ be next month's percentage change in its jet fuel cost and $Y_2$ the percentage change in the Brent crude price. The finance team models $(Y_1, Y_2)$ as bivariate normal with

$$\mu_1 = 1.8, \quad \sigma_1 = 7, \quad \mu_2 = 1.2, \quad \sigma_2 = 7, \quad \rho = 0.6.$$

Brent futures now indicate that the Brent change will be $y_2 = 11.7$ percent. Treat this as the observed value of $Y_2$.

**(a)** $E(Y_1 \mid Y_2 = 11.7) =$ ____________

**(b)** The conditional standard deviation of $Y_1$ given $Y_2 = 11.7$ is ____________

**(c)** Knowing Brent reduces the variance of the fuel-cost change from $\sigma_1^2$ to $V(Y_1 \mid Y_2 = y_2)$. By what percentage is the variance reduced? ____________ %

**(d)** Using Table 4, find the probability, given $Y_2 = 11.7$, that the fuel-cost change falls within 11.872 percentage points of its conditional mean, to four decimal places.

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## Problem 10

seed 20260925

In a household budget survey for Baku, let $Y_1$ be a household's monthly income and $Y_2$ its monthly consumption spending, both in AZN. The joint distribution is approximately bivariate normal with

$$\mu_1 = 1000, \quad \sigma_1 = 450, \quad \mu_2 = 730, \quad \sigma_2 = 315, \quad \rho = 0.74.$$

The function $y_1 \mapsto E(Y_2 \mid Y_1 = y_1)$ is called the **regression function** of consumption on income.

**(a)** Find it as a function of $y_1$: $E(Y_2 \mid Y_1 = y_1) =$ ____________

(enter an expression in $y_1$, typed as y_1)

**(b)** Its slope, the extra expected consumption per additional AZN of income (the marginal propensity to consume):

____________

**(c)** The expected consumption of a household with income 1765 AZN: ____________ AZN

**(d)** The standard deviation of consumption among households with that income: ____________ AZN

## Problem 11

seed 20260925

A bank offers business credit lines. For a randomly chosen client let $Y_2$ be the approved limit and $Y_1$ the balance actually drawn at month-end, both in thousand AZN. The drawn balance never exceeds the limit, and the joint density is

$$f(y_1, y_2) = \begin{cases} \dfrac{6\,y_1}{38^3}, & 0 \le y_1 \le y_2 \le 38, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** The marginal density of the limit, for $0 \le y_2 \le 38$: $f_2(y_2) =$ ______________

**(b)** $E(Y_1 \mid Y_2 = y_2) =$ ______________ for $0 < y_2 \le 38$

(enter expressions in $y_2$, typed as y_2; powers like 38^3 may be left unevaluated)

**(c)** For a client with a limit of 19 thousand AZN, the expected drawn balance is ______________ thousand AZN.

**(d)** Use Theorem 5.14 to find $E(Y_1) =$ ______________ thousand AZN.

## Problem 12

seed 20260925

A petrol station on the Baku-Shamakhi road receives one fuel delivery a week. The delivered volume $Y_1$ (thousand litres) is uniformly distributed on $(25, 68)$. Given $Y_1 = y_1$, the volume $Y_2$ sold before the next delivery is uniformly distributed on $(0, y_1)$.

**(a)** $E(Y_2 \mid Y_1 = y_1) =$ ______________ for $25 < y_1 < 68$

(enter an expression in $y_1$, typed as y_1)

**(b)** The expected weekly sales, $E(Y_2) =$ ______________ thousand litres

**(c)** $E[V(Y_2 \mid Y_1)] =$ ______________

**(d)** $V(Y_2) =$ ______________

Give (c) and (d) to at least four significant figures, in (thousand litres)$^2$.

## Problem 13

seed 20260925

A mobile operator in Azerbaijan follows a monthly panel of 28 prepaid customers. Given the month's churn probability $p$, the number $Y$ of panel customers who leave is binomial with parameters $n = 28$ and $p$. Competitors' promotions make $p$ vary from month to month. The operator models it as uniform on $(0.1, 0.21)$.

Give every answer to at least four significant figures.

**(a)** $E(Y) =$ ______________

**(b)** $V(Y) =$ ______________

**(c)** An analyst ignores the month-to-month variation and uses a binomial distribution with $p$ fixed at $E(p)$. What variance does that model give for $Y$? ______________

## Problem 14

seed 20260925

A private health insurer in Baku models the number $Y$ of outpatient claims a policyholder files in a year. Given the policyholder's claim rate $\lambda$, $Y$ is Poisson with mean $\lambda$. Rates differ between policyholders, and across the book $\lambda$ has a gamma distribution with $\alpha = 2$ and $\beta = 0.76$.

Give every answer to at least four significant figures.

**(a)** For a randomly chosen policyholder, $E(Y) =$ ______________

**(b)** $V(Y) =$ ______________

**(c)** A Poisson model with the same mean would have $V(Y) = E(Y)$. Find the ratio $V(Y)/E(Y)$ for the gamma-mixed model: ______________

**(d)** Use Tchebysheff's theorem to give an upper bound for the probability that the number of claims differs from $E(Y)$ by 5 or more: $P(|Y - E(Y)| \geq 5) \leq$ ______________

## Problem 15

seed 20260925

An insurer's property book for small businesses in Sumgait produces $N$ claims a month, where $N$ is Poisson with mean $\lambda = 28$. Claim sizes $X_1, X_2, \ldots$ (thousand AZN) are independent of each other and of $N$, each with mean $\mu = 5.7$ and standard deviation $\sigma = 2.5$. The monthly total is $T = X_1 + \cdots + X_N$ ($T = 0$ if $N = 0$).

Give every answer to at least four significant figures.

**(a)** $E(T) =$ ______________ thousand AZN

**(b)** $E[V(T \mid N)] =$ ______________

**(c)** $V[E(T \mid N)] =$ ______________

**(d)** $V(T) =$ ______________ (thousand AZN)$^2$

## Problem 16

seed 20260925

A pension fund in Baku holds mostly local assets, so its annual return $R$ (in percent) depends on the oil-market regime $S$. The fund's risk team uses the following model.

| Regime $s$ | $P(S = s)$ | $E(R \mid S = s)$ | $\sqrt{V(R \mid S = s)}$ |
|---|---|---|---|
| oil slump | 0.13 | -5.5 | 17 |
| normal | 0.58 | 8.5 | 9 |
| oil boom | 0.29 | 14 | 11 |

Give every answer to at least four significant figures.

**(a)** $E(R) =$ ______________ %

**(b)** The average within-regime variance, $E[V(R \mid S)] =$ ____________

**(c)** The variance due to switching between regimes, $V[E(R \mid S)] =$ ____________

**(d)** $V(R) =$ ____________

**(e)** What percentage of $V(R)$ is due to regime switching? ____________ %

## Problem 17

seed 20260925

A bank's markets division has two trading desks. Their daily profits $Y_1$ and $Y_2$ (thousand AZN) are bivariate normal with

$$\mu_1 = 4, \quad \sigma_1 = 14, \quad \mu_2 = 9.5, \quad \sigma_2 = 9, \quad \rho = 0.45.$$

Risk management tracks the division total $U_1 = Y_1 + Y_2$ and the performance gap $U_2 = Y_1 - Y_2$. Linear functions of a bivariate normal pair are again bivariate normal (Exercise 5.130), so $(U_1, U_2)$ is bivariate normal.

**(a)** $V(U_1) =$ ____________

**(b)** $V(U_2) =$ ____________

**(c)** $\mathrm{Cov}(U_1, U_2) =$ ____________

**(d)** Are the total and the gap independent?

- ◯ Yes, U1 and U2 are independent
- ◯ No, U1 and U2 are dependent

**(e)** Find the expected division total given the gap, as a function of $u$: $E(U_1 \mid U_2 = u) =$ ____________

(enter an expression in u)

## Problem 18

seed 20260925

Each month the Central Bank of Azerbaijan cuts ($Y_2 = -1$), holds ($Y_2 = 0$) or raises ($Y_2 = 1$) its policy rate. Let $Y_1$ be the number of new corporate bond issues placed on the Baku Stock Exchange in the following month (0, 1 or 2 in this model). An analyst estimates the joint probability function $p(y_1, y_2)$:

| | $y_1 = 0$ | $y_1 = 1$ | $y_1 = 2$ |
|---|---|---|---|
| $y_2 = -1$ (cut) | 0.038 | 0.038 | 0.114 |
| $y_2 = 0$ (hold) | 0.141 | 0.141 | 0.188 |
| $y_2 = 1$ (hike) | 0.238 | 0.034 | 0.068 |

Give every answer to at least four significant figures.

**(a)** The regression function of $Y_1$ on $Y_2$:

$E(Y_1 \mid Y_2 = -1) =$ ____________, $E(Y_1 \mid Y_2 = 0) =$ ____________, $E(Y_1 \mid Y_2 = 1) =$

____________

**(b)** $E(Y_1) =$ ______________

**(c)** $E[V(Y_1 \mid Y_2)] =$ ______________

**(d)** $V[E(Y_1 \mid Y_2)] =$ ______________

**(e)** $V(Y_1) =$ ______________

## Problem 19

seed 20260925

A state-supported agricultural insurer covers wheat farms in the Shirvan plain against drought damage. A season is dry with probability 0.2 and normal otherwise. Given a dry season, the number $N$ of claims from the district is Poisson with mean 19. Given a normal season, it is Poisson with mean 3.5.

Claim sizes $X_1, X_2, \ldots$ (thousand AZN) are exponential with mean $\beta = 9$, independent of each other, of $N$ and of the season. The district's total payout is $T = X_1 + \cdots + X_N$ ($T = 0$ if $N = 0$).

Give every answer to at least four significant figures.

**(a)** $E(N) =$ ______________

**(b)** $V(N) =$ ______________

**(c)** $E(T) =$ ______________ thousand AZN

**(d)** $V(T) =$ ______________ (thousand AZN)$^2$

## Problem 20

seed 20260925

A development bank lends to small and medium-sized enterprises through regional branches. At a branch with 18 SME loans, the number $Y$ of defaults in a year is binomial with parameters $n = 18$ and $p$, given that branch's default probability $p$. Across branches, $p$ has a beta distribution with $\alpha = 1$ and $\beta = 38$. A branch is chosen at random.

Give every answer to at least four significant figures, as a decimal number.

**(a)** $E(p) =$ ______________

**(b)** $E(Y) =$ ______________

**(c)** $V(Y) =$ ______________

**(d)** The probability that the branch has no defaults at all, $P(Y = 0) =$ ______________

(Hint: condition on $p$ and use the beta density. The answer is a product of 18 simple fractions.)