Week 15, Problem Set 1 - worked solutions

Wackerly 5.10-5.11 - The Bivariate Normal Distribution and Conditional Expectations

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A Baku trading company invoices partners in Russia and Georgia. Let Y_1 be next month's percentage change in the ruble per US dollar and Y_2 the percentage change in the lari per US dollar. The treasury desk models (Y_1, Y_2) as bivariate normal (Section 5.10) with

$$\mu_1 = -0.5, \quad \sigma_1 = 8, \quad \mu_2 = 0, \quad \sigma_2 = 5.5, \quad \rho = 0.6.$$

(a) The marginal distribution of Y_2 is normal. Its mean is ______________ and its standard deviation is ______________.

(b) $\text{Cov}(Y_1, Y_2) =$ ______________

(c) Using Table 4, find the probability that the lari weakens by more than 11.55 percent next month, $P(Y_2 > 11.55)$, to four decimal places.

____

(a) The bivariate normal density has five parameters, and the notation is not a coincidence: integrating y_1 out of $f(y_1, y_2)$ (Exercise 5.128) leaves a normal density with mean μ_2 and variance σ_2^2. So Y_2 is normal with mean 0 and standard deviation 5.5. The correlation ρ plays no part in the marginal.

(b) For the bivariate normal, $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2$, so

$$\text{Cov}(Y_1, Y_2) = 0.6 \times 8 \times 5.5 = 26.4.$$

(c) Only the marginal of Y_2 is needed. Standardise:

$$P(Y_2 > 11.55) = P\left(Z > \frac{11.55 - (0)}{5.5}\right) = P(Z > 2.10) = 0.0179.$$

The ruble's behaviour does not change this unconditional probability. It would matter only if we were told the ruble's move first. That is a conditional probability, which uses the conditional distribution of Y_2 given Y_1.

Problem 2

seed 20260925

A Baku bank issues standard ($Y_2 = 0$) and premium ($Y_2 = 1$) credit cards. For a randomly chosen cardholder let Y_1 be the number of late payments in the past year (the bank caps the count at 2). The joint probability function $p(y_1, y_2)$ is

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = 0$ (standard)	0.49	0.09	0.10
$y_2 = 1$ (premium)	0.21	0.08	0.03

Give every answer to at least four significant figures.

(a) $E(Y_1 \mid Y_2 = 0) =$ ______________

(b) $E(Y_1 \mid Y_2 = 1) =$ ______________

(c) Use Theorem 5.14 and your answers to (a) and (b) to find $E(Y_1)$: ______________

First the marginal of the card type: $P(Y_2 = 0) = 0.49 + 0.09 + 0.10 = 0.68$ and $P(Y_2 = 1) = 0.21 + 0.08 + 0.03 = 0.32$.

(a) The conditional probability function is $p(y_1 \mid 0) = p(y_1, 0)/P(Y_2 = 0)$, and Definition 5.13 averages y_1 against it:

$$E(Y_1 \mid Y_2 = 0) = \sum_{y_1} y_1 \frac{p(y_1, 0)}{P(Y_2 = 0)} = \frac{0(0.49) + 1(0.09) + 2(0.10)}{0.68} = \frac{0.29}{0.68} = 0.4265.$$

(b) In the same way,

$$E(Y_1 \mid Y_2 = 1) = \frac{0(0.21) + 1(0.08) + 2(0.03)}{0.32} = \frac{0.14}{0.32} = 0.4375.$$

So premium cardholders have more late payments on average than standard ones.

(c) $E(Y_1 \mid Y_2)$ is a random variable taking the value 0.4265 with probability 0.68 and 0.4375 with probability 0.32. Theorem 5.14 says its mean is $E(Y_1)$:

$$E(Y_1) = E[E(Y_1 \mid Y_2)] = 0.4265(0.68) + 0.4375(0.32) = 0.4300.$$

Check directly from the table: $E(Y_1) = 1(0.09 + 0.08) + 2(0.10 + 0.03) = 0.4300$. The two routes must agree, because the conditional means weighted by $P(Y_2 = y_2)$ just undo the division in (a) and (b).

Problem 3

seed 20260925

Each morning a bank loads Y_2 thousand AZN into a street ATM whose capacity is 54 thousand AZN, and Y_1 thousand AZN is withdrawn during the day. The withdrawals cannot exceed what was loaded. The joint density is

$$f(y_1, y_2) = \begin{cases} \frac{2}{2916}, & 0 \le y_1 \le y_2 \le 54, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the conditional expectation of the day's withdrawals given the amount loaded, as a function of y_2 (for $0 < y_2 \le 54$):

$E(Y_1 \mid Y_2 = y_2) =$ ______________

(enter an expression in y_2, typed as y_2)

(b) If 48.6 thousand AZN is loaded, the expected withdrawal is ______________ thousand AZN.

(c) Use Theorem 5.14 to find the long-run average daily withdrawal, $E(Y_1) =$ ______________ thousand AZN. Give it to at least four significant figures.

The marginal density of the loaded amount is, for $0 \le y_2 \le 54$,

$$f_2(y_2) = \int_0^{y_2} \frac{2}{2916}\, dy_1 = \frac{2y_2}{2916}.$$

(a) For $0 < y_2 \le 54$ the conditional density is

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{2/2916}{2y_2/2916} = \frac{1}{y_2}, \qquad 0 \le y_1 \le y_2,$$

a uniform density on $[0, y_2]$. By Definition 5.13,

$$E(Y_1 \mid Y_2 = y_2) = \int_0^{y_2} y_1 \cdot \frac{1}{y_2}\, dy_1 = \frac{1}{y_2} \cdot \frac{y_2^2}{2} = \frac{y_2}{2}.$$

(b) At $y_2 = 48.6$: $E(Y_1 \mid Y_2 = 48.6) = 48.6/2 = 24.3$ thousand AZN.

(c) Since $E(Y_1 \mid Y_2) = Y_2/2$, Theorem 5.14 gives $E(Y_1) = E(Y_2/2) = E(Y_2)/2$, and

$$E(Y_2) = \int_0^{54} y_2 \cdot \frac{2y_2}{2916}\, dy_2 = \frac{2}{2916} \cdot \frac{54^3}{3} = \frac{2(54)}{3} = 36.0000.$$

So $E(Y_1) = 36.0000/2 = 18.0000$ thousand AZN, one third of capacity. No double integral over the triangle was needed.

Problem 4

seed 20260925

A motor insurer in Baku classifies drivers as low-risk or high-risk. For a driver whose annual claim rate is λ, the number of claims Y in a year has a Poisson distribution with mean λ. The rate itself depends on the driver:

- with probability 0.6 the driver is low-risk and $\lambda = 0.19$;
- with probability 0.4 the driver is high-risk and $\lambda = 0.5$.

A policy is chosen at random from the book. Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________

(b) The probability that the policy has no claims in the year, $P(Y = 0) =$ ______________

(c) The insurer holds 5500 such policies. The expected total number of claims in the book is ______________

(a) Given the rate, Y is Poisson, so $E(Y \mid \lambda) = \lambda$. The rate is itself random, and Theorem 5.14 averages the conditional mean over it:

$$E(Y) = E[E(Y \mid \lambda)] = E(\lambda) = 0.6(0.19) + 0.4(0.5) = 0.3140.$$

(b) The same idea applies to a probability, because $P(Y = 0) = E[I(Y = 0)]$, where I is an indicator. Its conditional expectation is $P(Y = 0 \mid \lambda) = e^{-\lambda}$, so

$$P(Y = 0) = E\left(e^{-\lambda}\right) = 0.6\,e^{-0.19} + 0.4\,e^{-0.5} = 0.6(0.82696) + 0.4(0.60653) = 0.73879.$$

Note that this is **not** $e^{-E(\lambda)}$. Averaging $e^{-\lambda}$ and then plugging in the average rate give different answers.

(c) Expectation is linear. Each of the 5500 policies has mean 0.3140, so the book expects $5500 \times 0.3140 = 1727.00$ claims.

Problem 5

seed 20260925

An agricultural insurer covers orchards in the Guba district against hail. On a given policy the number of claims N in a season has probability function

n	0	1	2	3
$p(n)$	0.64	0.18	0.15	0.03

Each claim, whatever the number of claims, has mean 2300 AZN, and the claim sizes are independent of N. Let T be the total paid on the policy in a season ($T = 0$ when $N = 0$).

(a) $E(T \mid N = 2) =$ ______________ AZN

(b) $E(N) =$ ______________

(c) $E(T) =$ ______________ AZN

(a) Given $N = 2$, T is the sum of 2 claims, each with mean 2300. The expectation of a sum is the sum of the expectations (Theorem 5.12), so $E(T \mid N = 2) = 2(2300) = 4600$ AZN. In general $E(T \mid N = n) = n(2300)$, so $E(T \mid N) = 2300\,N$.

(b) $E(N) = 0(0.64) + 1(0.18) + 2(0.15) + 3(0.03) = 0.57$.

(c) By Theorem 5.14,

$$E(T) = E[E(T \mid N)] = E(2300\,N) = 2300\,E(N) = 2300(0.57) = 1311 \text{ AZN}.$$

This is the premium an insurer must at least charge on average: expected number of claims times expected claim size. It uses the independence of the claim sizes and N only through the fact that the mean claim is the same whatever N turns out to be.

Problem 6

seed 20260925

A microfinance lender in Ganja has 17 loans maturing each week. Given the week's default probability p, the number Y of those loans that default is binomial with parameters $n = 17$ and p (loans default independently). But p moves with local economic conditions from week to week, and the lender models it as uniformly distributed on the interval $(0.05, 0.12)$.

(a) Find $E(Y \mid p)$ as a function of p: ______________

(enter an expression in p)

(b) $E(p) =$ ______________

(c) The expected number of defaults in a week, $E(Y) =$ ______________

This is a hierarchical model, exactly as in Example 5.32. The distribution of Y is given conditionally on the parameter p, and p has a distribution of its own.

(a) For a fixed p, Y is binomial, so by Theorem 3.7 $E(Y \mid p) = np = 17p$.

(b) For a uniform distribution on (θ_1, θ_2) the mean is $(\theta_1 + \theta_2)/2$ (Theorem 4.6), so $E(p) = (0.05 + 0.12)/2 = 0.085$.

(c) By Theorem 5.14,

$$E(Y) = E[E(Y \mid p)] = E(17p) = 17\,E(p) = 17(0.085) = 1.445.$$

We never needed the unconditional probability function of Y, which would require integrating the binomial probabilities against the uniform density.

Problem 7

seed 20260925

The Ministry of Economy's trade unit models monthly percentage changes in export volumes. Each model below is a **bivariate normal** distribution.

Model I (hazelnuts X_1, persimmons X_2): $\mu_1 = 2.8$, $\sigma_1 = 6$, $\mu_2 = 1.7$, $\sigma_2 = 5$, $\mathrm{Cov}(X_1, X_2) = 0$.

Model II (cotton W_1, wine W_2): $\mu_1 = 2.2$, $\sigma_1 = 9$, $\mu_2 = 2.3$, $\sigma_2 = 2$, $\mathrm{Cov}(W_1, W_2) = -5.4$.

(a) In which model are the two export changes independent?

- ○ Model I only
- ○ Model II only
- ○ Both models
- ○ Neither model

(b) For the model you chose in (a), use Table 4 to find the probability that **both** export volumes grow by more than their thresholds next month, the first by more than 9.76 percent and the second by more than 8.9 percent. Give it to four decimal places.

____

(c) For the other model, find the correlation coefficient ρ of the two export changes: ____________

(a) In general, zero covariance does **not** imply independence. For a bivariate normal pair it does. When $\rho = 0$ the quadratic form Q in the exponent loses its cross term, and the density factors as $g(y_1)h(y_2)$. Theorem 5.5 then gives independence. Conversely, independent variables have zero covariance. So a bivariate normal pair is independent exactly when its covariance is 0.

Model I has covariance 0, so its two variables are independent. Model II has covariance $-5.4 \neq 0$, so its variables are dependent. The answer is Model I only.

(b) In Model I (hazelnut X_1, persimmon X_2), independence lets the joint probability factor into two marginal probabilities, each from a normal marginal:

$$P(X_1 > 9.76) = P\left(Z > \frac{9.76 - (2.8)}{6}\right) = P(Z > 1.16) = 0.1230,$$

$$P(X_2 > 8.9) = P\left(Z > \frac{8.9 - (1.7)}{5}\right) = P(Z > 1.44) = 0.0749,$$

$$P(X_1 > 9.76,\ X_2 > 8.9) = 0.1230 \times 0.0749 = 0.0092.$$

The same product would be wrong for Model II. There the joint tail needs the bivariate normal density itself.

(c) For the bivariate normal, $\mathrm{Cov} = \rho\sigma_1\sigma_2$, so in Model II

$$\rho = \frac{\mathrm{Cov}(W_1, W_2)}{\sigma_1\sigma_2} = \frac{-5.4}{9 \times 2} = -0.3.$$

Problem 8

seed 20260925

Azerbaijan imports most of its milling wheat. Let Y_1 be next month's percentage change in the world wheat price and Y_2 the percentage change in the Baku bread price index. An analyst at the central bank treats (Y_1, Y_2) as bivariate normal with

$$\mu_1 = 2.8, \quad \sigma_1 = 5, \quad \mu_2 = 1.4, \quad \sigma_2 = 2, \quad \rho = 0.28.$$

The wheat price turns out to have changed by $y_1 = 10.3$ percent.

(a) $E(Y_2 \mid Y_1 = 10.3) =$ ______________

(b) $V(Y_2 \mid Y_1 = 10.3) =$ ______________

(c) Using Table 4, find the conditional probability that bread prices rise by more than 5.1968 percent, $P(Y_2 > 5.1968 \mid Y_1 = 10.3)$, to four decimal places.

____

Given $Y_1 = y_1$, Y_2 is normal (Exercise 5.129, with the roles of the two variables exchanged), with

$$E(Y_2 \mid Y_1 = y_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(y_1 - \mu_1), \qquad V(Y_2 \mid Y_1 = y_1) = \sigma_2^2(1-\rho^2).$$

(a) The slope is $\rho\sigma_2/\sigma_1 = 0.28(2)/5 = 0.112$, and $y_1 - \mu_1 = 10.3 - (2.8) = 7.5$, so

$$E(Y_2 \mid Y_1 = 10.3) = 1.4 + 0.112 \times (7.5) = 2.24.$$

(b) $V(Y_2 \mid Y_1 = 10.3) = (2)^2(1 - 0.28^2) = (2)^2(0.9216) = 3.6864$. The conditional variance does not depend on the observed y_1. Seeing the wheat price removes the fraction ρ^2 of the bread-price variance, wherever the wheat price lands.

(c) The conditional standard deviation is $\sqrt{3.6864} = 1.92$. Standardise with the **conditional** mean and standard deviation:

$$P(Y_2 > 5.1968 \mid Y_1 = 10.3) = P\left(Z > \frac{5.1968 - (2.24)}{1.92}\right) = P(Z > 1.54) = 0.0618.$$

Problem 9

seed 20260925

A Baku-based airline budgets for fuel. Let Y_1 be next month's percentage change in its jet fuel cost and Y_2 the percentage change in the Brent crude price. The finance team models (Y_1, Y_2) as bivariate normal with

$$\mu_1 = 1.8, \quad \sigma_1 = 7, \quad \mu_2 = 1.2, \quad \sigma_2 = 7, \quad \rho = 0.6.$$

Brent futures now indicate that the Brent change will be $y_2 = 11.7$ percent. Treat this as the observed value of Y_2.

(a) $E(Y_1 \mid Y_2 = 11.7) =$ ______________

(b) The conditional standard deviation of Y_1 given $Y_2 = 11.7$ is ______________

(c) Knowing Brent reduces the variance of the fuel-cost change from σ_1^2 to $V(Y_1 \mid Y_2 = y_2)$. By what percentage is the variance reduced?

______________ %

(d) Using Table 4, find the probability, given $Y_2 = 11.7$, that the fuel-cost change falls within 11.872 percentage points of its conditional mean, to four decimal places.

____

By Exercise 5.129, given $Y_2 = y_2$ the variable Y_1 is normal with mean $\mu_1 + \rho(\sigma_1/\sigma_2)(y_2 - \mu_2)$ and variance $\sigma_1^2(1-\rho^2)$.

(a) The slope is $\rho\sigma_1/\sigma_2 = 0.6(7)/7 = 0.6$ and $y_2 - \mu_2 = 11.7 - (1.2) = 10.5$, so

$$E(Y_1 \mid Y_2 = 11.7) = 1.8 + 0.6 \times (10.5) = 8.1.$$

(b) $V(Y_1 \mid Y_2 = 11.7) = 49(1 - 0.6^2) = 49(0.64) = 31.36$, so the conditional standard deviation is $\sqrt{31.36} = 5.6$.

(c) The variance falls from $\sigma_1^2 = 49$ to $\sigma_1^2(1-\rho^2)$, a proportional reduction of $\rho^2 = 0.6^2$, that is 36%. It is the same whatever value of Brent is observed. That share of the risk is what a Brent hedge can remove. The remaining $(1-\rho^2)$ share is basis risk.

(d) The half-width is $11.872/5.6 = 2.12$ conditional standard deviations, so

$$P\left(|Y_1 - (8.1)| \le 11.872 \,|\, Y_2 = 11.7\right) = P(-2.12 \le Z \le 2.12) = 1 - 2A(2.12) = 1 - 2(0.0170) = 0.9660,$$

where $A(z) = P(Z > z)$ is the Table 4 area.

Problem 10

seed 20260925

In a household budget survey for Baku, let Y_1 be a household's monthly income and Y_2 its monthly consumption spending, both in AZN. The joint distribution is approximately bivariate normal with

$$\mu_1 = 1000, \quad \sigma_1 = 450, \quad \mu_2 = 730, \quad \sigma_2 = 315, \quad \rho = 0.74.$$

The function $y_1 \mapsto E(Y_2 \mid Y_1 = y_1)$ is called the **regression function** of consumption on income.

(a) Find it as a function of y_1: $E(Y_2 \mid Y_1 = y_1) =$ ____________

(enter an expression in y_1, typed as y_1)

(b) Its slope, the extra expected consumption per additional AZN of income (the marginal propensity to consume): ____________

(c) The expected consumption of a household with income 1765 AZN: ____________ AZN

(d) The standard deviation of consumption among households with that income: ____________ AZN

For a bivariate normal pair, the conditional distribution of Y_2 given $Y_1 = y_1$ is normal with mean $\mu_2 + \rho(\sigma_2/\sigma_1)(y_1 - \mu_1)$ and variance $\sigma_2^2(1-\rho^2)$ (Exercise 5.129 with the roles exchanged).

(a) The slope is $\rho\sigma_2/\sigma_1 = 0.74(315)/450 = 0.518$, so

$$E(Y_2 \mid Y_1 = y_1) = 730 + 0.518(y_1 - 1000) = 212 + 0.518\, y_1.$$

For the bivariate normal the regression function is a straight line. That is one reason the linear consumption function of introductory macroeconomics fits so naturally.

(b) The slope is 0.518. Each extra AZN of income raises expected consumption by about 0.518 AZN.

(c) $E(Y_2 \mid Y_1 = 1765) = 730 + 0.518(765) = 1126.27$ AZN.

(d) $\sqrt{V(Y_2 \mid Y_1 = y_1)} = \sigma_2\sqrt{1-\rho^2} = 315\sqrt{0.4524} = 211.8712$ AZN. This is the same at every income level. Also, by Theorem 5.14, $E(Y_2) = E[E(Y_2 \mid Y_1)] = 730 + 0.518(E(Y_1) - 1000) = 730$, as it must be.

Problem 11

seed 20260925

A bank offers business credit lines. For a randomly chosen client let Y_2 be the approved limit and Y_1 the balance actually drawn at month-end, both in thousand AZN. The drawn balance never exceeds the limit, and the joint density is

$$f(y_1, y_2) = \begin{cases} \dfrac{6\,y_1}{38^3}, & 0 \le y_1 \le y_2 \le 38, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) The marginal density of the limit, for $0 \le y_2 \le 38$: $f_2(y_2) =$ ____________

(b) $E(Y_1 \mid Y_2 = y_2) =$ ____________ for $0 < y_2 \le 38$

(enter expressions in y_2, typed as y_2; powers like 38^3 may be left unevaluated)

(c) For a client with a limit of 19 thousand AZN, the expected drawn balance is ____________ thousand AZN.

(d) Use Theorem 5.14 to find $E(Y_1) =$ ____________ thousand AZN.

(a) Integrate y_1 out over $0 \le y_1 \le y_2$:

$$f_2(y_2) = \int_0^{y_2} \frac{6\, y_1^1}{38^3}\, dy_1 = \frac{6}{38^3} \cdot \frac{y_2^2}{2} = \frac{3\, y_2^2}{38^3}, \qquad 0 \le y_2 \le 38.$$

(b) For $0 < y_2 \le 38$ the conditional density is

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{2\, y_1^1}{y_2^2}, \qquad 0 \le y_1 \le y_2,$$

and by Definition 5.13

$$E(Y_1 \mid Y_2 = y_2) = \int_0^{y_2} y_1 \cdot \frac{2\, y_1^1}{y_2^2}\, dy_1 = \frac{2}{y_2^2} \cdot \frac{y_2^3}{3} = \frac{2}{3}\, y_2.$$

So on average clients draw the fraction $2/3$ of their limit, whatever the limit is.

(c) $E(Y_1 \mid Y_2 = 19) = (2/3)(19) = 12.6667$.

(d) By Theorem 5.14, $E(Y_1) = E\left[\frac{2}{3} Y_2\right] = \frac{2}{3} E(Y_2)$, and

$$E(Y_2) = \int_0^{38} y_2 \cdot \frac{3\, y_2^2}{38^3}\, dy_2 = \frac{3}{4}(38) = 28.5.$$

Hence $E(Y_1) = (2/3)(28.5) = 19$ thousand AZN.

Problem 12

seed 20260925

A petrol station on the Baku-Shamakhi road receives one fuel delivery a week. The delivered volume Y_1 (thousand litres) is uniformly distributed on $(25, 68)$. Given $Y_1 = y_1$, the volume Y_2 sold before the next delivery is uniformly distributed on $(0, y_1)$.

(a) $E(Y_2 \mid Y_1 = y_1) =$ ____________ for $25 < y_1 < 68$

(enter an expression in y_1, typed as y_1)

(b) The expected weekly sales, $E(Y_2) =$ ____________ thousand litres

(c) $E[V(Y_2 \mid Y_1)] =$ ____________

(d) $V(Y_2) =$ ____________

Give (c) and (d) to at least four significant figures, in (thousand litres)^2.

Only the conditional distribution of Y_2 and the marginal of Y_1 are given. Theorems 5.14 and 5.15 let us avoid finding the marginal of Y_2.

(a) A uniform distribution on $(0, y_1)$ has mean $y_1/2$ and variance $y_1^2/12$ (Theorem 4.6). So $E(Y_2 \mid Y_1 = y_1) = y_1/2$ and $V(Y_2 \mid Y_1 = y_1) = y_1^2/12$.

(b) The marginal of Y_1 has $E(Y_1) = (25 + 68)/2 = 46.5$, and by Theorem 5.14

$$E(Y_2) = E[E(Y_2 \mid Y_1)] = E(Y_1/2) = 46.5/2 = 23.25.$$

(c) $V(Y_1) = (68 - 25)^2/12 = 154.0833$, so $E(Y_1^2) = V(Y_1) + [E(Y_1)]^2 = 154.0833 + 46.5^2 = 2316.3333$. Then

$$E[V(Y_2 \mid Y_1)] = E\left(\frac{Y_1^2}{12}\right) = \frac{2316.3333}{12} = 193.0278.$$

(d) $V[E(Y_2 \mid Y_1)] = V(Y_1/2) = V(Y_1)/4 = 154.0833/4 = 38.5208$, and by Theorem 5.15

$$V(Y_2) = E[V(Y_2 \mid Y_1)] + V[E(Y_2 \mid Y_1)] = 193.0278 + 38.5208 = 231.5486.$$

The standard deviation of weekly sales is 15.2167 thousand litres. Most of the variance comes from the first term: sales vary a lot even when the delivery size is known.

Problem 13

seed 20260925

A mobile operator in Azerbaijan follows a monthly panel of 28 prepaid customers. Given the month's churn probability p, the number Y of panel customers who leave is binomial with parameters $n = 28$ and p. Competitors' promotions make p vary from month to month. The operator models it as uniform on $(0.1, 0.21)$.

Give every answer to at least four significant figures.

(a) $E(Y) =$ ______________

(b) $V(Y) =$ ______________

(c) An analyst ignores the month-to-month variation and uses a binomial distribution with p fixed at $E(p)$. What variance does that model give for Y? ______________

For a fixed p, $E(Y \mid p) = np$ and $V(Y \mid p) = np(1-p)$ (Theorem 3.7). For the uniform p (Theorem 4.6): $E(p) = (0.1 + 0.21)/2 = 0.155$, $V(p) = (0.21 - 0.1)^2/12 = 0.00100833$, and $E(p^2) = V(p) + [E(p)]^2 = 0.02503333$.

(a) Theorem 5.14: $E(Y) = E(np) = 28(0.155) = 4.34$.

(b) Theorem 5.15, exactly as in Example 5.33:

$$E[V(Y \mid p)] = E[np(1-p)] = n[E(p) - E(p^2)] = 28(0.155 - 0.02503333) = 3.639067,$$

$$V[E(Y \mid p)] = V(np) = n^2 V(p) = 784(0.00100833) = 0.790533,$$

$$V(Y) = 3.639067 + 0.790533 = 4.4296.$$

(c) $nE(p)[1 - E(p)] = 28(0.155)(1 - 0.155) = 3.6673$.

The fixed-p model understates the variance by 0.7623. Algebraically the gap is $n(n-1)V(p) = 756(0.00100833)$. This is overdispersion: a random churn rate makes very good and very bad months more likely than a binomial with the average rate would suggest.

Problem 14

seed 20260925

A private health insurer in Baku models the number Y of outpatient claims a policyholder files in a year. Given the policyholder's claim rate λ, Y is Poisson with mean λ. Rates differ between policyholders, and across the book λ has a gamma distribution with $\alpha = 2$ and $\beta = 0.76$.

Give every answer to at least four significant figures.

(a) For a randomly chosen policyholder, $E(Y) =$ ______________

(b) $V(Y) =$ ______________

(c) A Poisson model with the same mean would have $V(Y) = E(Y)$. Find the ratio $V(Y)/E(Y)$ for the gamma-mixed model: ______________

(d) Use Tchebysheff's theorem to give an upper bound for the probability that the number of claims differs from $E(Y)$ by 5 or more: $P(|Y - E(Y)| \geq 5) \leq$ ______________

Given λ, $E(Y \mid \lambda) = \lambda$ and $V(Y \mid \lambda) = \lambda$ (Theorem 3.11). For the gamma rate (Theorem 4.8), $E(\lambda) = \alpha\beta = 2(0.76) = 1.52$ and $V(\lambda) = \alpha\beta^2 = 2(0.76)^2 = 1.1552$.

(a) Theorem 5.14: $E(Y) = E[E(Y \mid \lambda)] = E(\lambda) = 1.52$.

(b) Theorem 5.15:

$$V(Y) = E[V(Y \mid \lambda)] + V[E(Y \mid \lambda)] = E(\lambda) + V(\lambda) = 1.52 + 1.1552 = 2.6752.$$

(c) $V(Y)/E(Y) = \alpha\beta(1+\beta)/(\alpha\beta) = 1 + \beta = 1.76$. The first term of Theorem 5.15 is the Poisson noise every policyholder has. The second term is the spread of risk between policyholders, and it makes the count overdispersed.

(d) The standard deviation is $\sigma = \sqrt{2.6752} = 1.6356$. Tchebysheff's theorem (Theorem 3.14) with $k\sigma = 5$ gives $P(|Y - \mu| \geq k\sigma) \leq 1/k^2 = \sigma^2/5^2$:

$$P(|Y - E(Y)| \geq 5) \leq \frac{2.6752}{25} = 0.107008.$$

The bound uses only the mean and variance, which Theorems 5.14 and 5.15 gave without our finding the distribution of Y.

Problem 15

seed 20260925

An insurer's property book for small businesses in Sumgait produces N claims a month, where N is Poisson with mean $\lambda = 28$. Claim sizes $X_1, X_2, \ldots$ (thousand AZN) are independent of each other and of N, each with mean $\mu = 5.7$ and standard deviation $\sigma = 2.5$. The monthly total is $T = X_1 + \cdots + X_N$ ($T = 0$ if $N = 0$).

Give every answer to at least four significant figures.

(a) $E(T) =$ ____________ thousand AZN

(b) $E[V(T \mid N)] =$ ____________

(c) $V[E(T \mid N)] =$ ____________

(d) $V(T) =$ ____________ (thousand AZN)^2

Condition on the number of claims. Given $N = n$, T is a sum of n independent claims, so by Theorem 5.12

$$E(T \mid N = n) = n\mu, \qquad V(T \mid N = n) = n\sigma^2,$$

that is, $E(T \mid N) = \mu N$ and $V(T \mid N) = \sigma^2 N$. For the Poisson count, $E(N) = V(N) = 28$ (Theorem 3.11).

(a) Theorem 5.14: $E(T) = E(\mu N) = \mu E(N) = 5.7(28) = 159.6$.

(b) $E[V(T \mid N)] = E(\sigma^2 N) = \sigma^2 E(N) = 6.25(28) = 175$.

(c) $V[E(T \mid N)] = V(\mu N) = \mu^2 V(N) = 32.49(28) = 909.72$.

(d) Theorem 5.15:

$$V(T) = E[V(T \mid N)] + V[E(T \mid N)] = 175 + 909.72 = 1084.72,$$

so the standard deviation of the monthly total is 32.9351 thousand AZN. The two terms separate the two sources of risk: (b) is uncertainty about claim **sizes**, and (c) is uncertainty about the **number** of claims. For a compound Poisson sum they combine to $\lambda(\sigma^2 + \mu^2) = \lambda E(X^2)$.

Problem 16

seed 20260925

A pension fund in Baku holds mostly local assets, so its annual return R (in percent) depends on the oil-market regime S. The fund's risk team uses the following model.

Regime s	$P(S = s)$	$E(R \mid S = s)$	$\sqrt{V(R \mid S = s)}$
oil slump	0.13	-5.5	17
normal	0.58	8.5	9
oil boom	0.29	14	11

Give every answer to at least four significant figures.

(a) $E(R) =$ ____________ %

(b) The average within-regime variance, $E[V(R \mid S)] =$ ____________

(c) The variance due to switching between regimes, $V[E(R \mid S)] =$ ____________

(d) $V(R) =$ ____________

(e) What percentage of $V(R)$ is due to regime switching? ____________ %

$E(R \mid S)$ and $V(R \mid S)$ are functions of the random regime S, so each is a random variable with the three values in the table.

(a) Theorem 5.14: $E(R) = E[E(R \mid S)] = 0.13(-5.5) + 0.58(8.5) + 0.29(14) = 8.275$.

(b) The conditional variances are the squared standard deviations $289, 81, 121$:

$$E[V(R \mid S)] = 0.13(289) + 0.58(81) + 0.29(121) = 119.64.$$

(c) Use $V(U) = E(U^2) - [E(U)]^2$ with $U = E(R \mid S)$:

$$E\{[E(R \mid S)]^2\} = 0.13(-5.5)^2 + 0.58(8.5)^2 + 0.29(14)^2 = 102.6775,$$

$$V[E(R \mid S)] = 102.6775 - (8.275)^2 = 34.2019.$$

(d) Theorem 5.15: $V(R) = E[V(R \mid S)] + V[E(R \mid S)] = 119.64 + 34.2019 = 153.8419$, a standard deviation of 12.4033%.

(e) $100 \times 34.2019/153.8419 = 22.23$%. That share of the fund's risk comes from not knowing **which** regime the year will bring. The rest is ordinary market noise within a regime. Diversifying inside the local market cannot remove the regime share. Only assets whose returns do not depend on oil can.

Problem 17

seed 20260925

A bank's markets division has two trading desks. Their daily profits Y_1 and Y_2 (thousand AZN) are bivariate normal with

$$\mu_1 = 4, \quad \sigma_1 = 14, \quad \mu_2 = 9.5, \quad \sigma_2 = 9, \quad \rho = 0.45.$$

Risk management tracks the division total $U_1 = Y_1 + Y_2$ and the performance gap $U_2 = Y_1 - Y_2$. Linear functions of a bivariate normal pair are again bivariate normal (Exercise 5.130), so (U_1, U_2) is bivariate normal.

(a) $V(U_1) =$ ____________

(b) $V(U_2) =$ ____________

(c) $\text{Cov}(U_1, U_2) =$ ____________

(d) Are the total and the gap independent?

- ○ Yes, U1 and U2 are independent
- ○ No, U1 and U2 are dependent

(e) Find the expected division total given the gap, as a function of u: $E(U_1 \mid U_2 = u) =$ ____________

(enter an expression in u)

First, $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = 0.45(14)(9) = 56.7$.

(a) By Theorem 5.12, $V(U_1) = \sigma_1^2 + \sigma_2^2 + 2\text{Cov}(Y_1, Y_2) = 196 + 81 + (113.4) = 390.4$.

(b) $V(U_2) = \sigma_1^2 + \sigma_2^2 - 2\text{Cov}(Y_1, Y_2) = 196 + 81 - (113.4) = 163.6$.

(c) Again by Theorem 5.12,

$$\text{Cov}(Y_1 + Y_2,\, Y_1 - Y_2) = V(Y_1) - \text{Cov}(Y_1, Y_2) + \text{Cov}(Y_2, Y_1) - V(Y_2) = \sigma_1^2 - \sigma_2^2 = 196 - 81 = 115.$$

The correlation ρ cancels completely.

(d) (U_1, U_2) is bivariate normal, and a bivariate normal pair is independent exactly when its covariance is zero (Section 5.10). By (c) that happens exactly when $\sigma_1 = \sigma_2$, whatever ρ is. Here the two standard deviations differ, so the covariance is not 0 and the total and the gap are dependent.

(e) For a bivariate normal pair, $E(U_1 \mid U_2 = u) = \mu_{U_1} + \rho_U \frac{\sigma_{U_1}}{\sigma_{U_2}}(u - \mu_{U_2})$, and $\rho_U\sigma_{U_1}/\sigma_{U_2} = \text{Cov}(U_1, U_2)/V(U_2)$. With $\mu_{U_1} = \mu_1 + \mu_2 = 13.5$ and $\mu_{U_2} = \mu_1 - \mu_2 = -5.5$,

$$E(U_1 \mid U_2 = u) = 13.5 + \frac{115}{163.6}\,(u - (-5.5)) = 13.5 + 0.702934\,(u - (-5.5)).$$

When the desks are equally volatile, the slope is 0. Knowing the gap then tells you nothing about the total.

Problem 18

seed 20260925

Each month the Central Bank of Azerbaijan cuts ($Y_2 = -1$), holds ($Y_2 = 0$) or raises ($Y_2 = 1$) its policy rate. Let Y_1 be the number of new corporate bond issues placed on the Baku Stock Exchange in the following month (0, 1 or 2 in this model). An analyst estimates the joint probability function $p(y_1, y_2)$:

	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$
$y_2 = -1$ (cut)	0.038	0.038	0.114
$y_2 = 0$ (hold)	0.141	0.141	0.188
$y_2 = 1$ (hike)	0.238	0.034	0.068

Give every answer to at least four significant figures.

(a) The regression function of Y_1 on Y_2:

$E(Y_1 \mid Y_2 = -1) =$ __________, $E(Y_1 \mid Y_2 = 0) =$ __________, $E(Y_1 \mid Y_2 = 1) =$ __________

(b) $E(Y_1) =$ __________

(c) $E[V(Y_1 \mid Y_2)] =$ __________

(d) $V[E(Y_1 \mid Y_2)] =$ __________

(e) $V(Y_1) =$ __________

Row sums give the marginal of the decision: $P(Y_2 = -1) = 0.19$, $P(Y_2 = 0) = 0.47$, $P(Y_2 = 1) = 0.34$. Dividing each row by its sum gives $p(y_1 \mid y_2)$.

(a) By Definition 5.13, $E(Y_1 \mid Y_2 = y_2) = \sum_{y_1} y_1\, p(y_1, y_2)/P(Y_2 = y_2)$:

$$E(Y_1 \mid Y_2 = -1) = \frac{0.038 + 2(0.114)}{0.19} = 1.4, \quad E(Y_1 \mid Y_2 = 0) = \frac{0.141 + 2(0.188)}{0.47} = 1.1, \quad E(Y_1 \mid Y_2 = 1) = \frac{0.034 + 2(0.068)}{0.34} = 0.5.$$

(b) Theorem 5.14: $E(Y_1) = 0.19(1.4) + 0.47(1.1) + 0.34(0.5) = 0.953$.

(c) The conditional second moments are $E(Y_1^2 \mid Y_2 = y_2) = [p(1, y_2) + 4p(2, y_2)]/P(Y_2 = y_2)$, namely 2.6, 1.9 and 0.9. Subtracting the squared conditional means gives the conditional variances $V(Y_1 \mid Y_2 = -1) = 0.64$, $V(Y_1 \mid Y_2 = 0) = 0.69$, $V(Y_1 \mid Y_2 = 1) = 0.65$. Hence

$$E[V(Y_1 \mid Y_2)] = 0.19(0.64) + 0.47(0.69) + 0.34(0.65) = 0.6669.$$

(d) $E\{[E(Y_1 \mid Y_2)]^2\} = 0.19(1.4)^2 + 0.47(1.1)^2 + 0.34(0.5)^2 = 1.0261$, so

$$V[E(Y_1 \mid Y_2)] = 1.0261 - (0.953)^2 = 0.117891.$$

(e) Theorem 5.15: $V(Y_1) = 0.6669 + 0.117891 = 0.784791$.

Check directly: $E(Y_1^2) = 1.693$ from the table, and $V(Y_1) = 1.693 - (0.953)^2 = 0.784791$. The rate decision accounts for the part 0.117891 of the variance in issuance. The rest is variation that remains even after the decision is known.

Problem 19

seed 20260925

A state-supported agricultural insurer covers wheat farms in the Shirvan plain against drought damage. A season is dry with probability 0.2 and normal otherwise. Given a dry season, the number N of claims from the district is Poisson with mean 19. Given a normal season, it is Poisson with mean 3.5.

Claim sizes $X_1, X_2, \ldots$ (thousand AZN) are exponential with mean $\beta = 9$, independent of each other, of N and of the season. The district's total payout is $T = X_1 + \cdots + X_N$ ($T = 0$ if $N = 0$).

Give every answer to at least four significant figures.

(a) $E(N) =$ __________

(b) $V(N) =$ __________

(c) $E(T) =$ ____________ thousand AZN

(d) $V(T) =$ ____________ (thousand AZN)^2

Layer 1: the count. Let λ be the season's claim rate, equal to 19 with probability 0.2 and 3.5 with probability 0.8. Given λ, N is Poisson, so $E(N \mid \lambda) = V(N \mid \lambda) = \lambda$ (Theorem 3.11).

(a) Theorem 5.14: $E(N) = E(\lambda) = 0.2(19) + 0.8(3.5) = 6.6$.

(b) λ is a two-point variable, so $V(\lambda) = q(1-q)(\lambda_D - \lambda_N)^2 = 0.2(0.8)(15.5)^2 = 38.44$. By Theorem 5.15

$$V(N) = E[V(N \mid \lambda)] + V[E(N \mid \lambda)] = E(\lambda) + V(\lambda) = 6.6 + 38.44 = 45.04.$$

The count is overdispersed: $V(N) > E(N)$, unlike a single Poisson.

Layer 2: the total. Given $N = n$, T is a sum of n independent exponentials, so $E(T \mid N) = \beta N$ and $V(T \mid N) = \beta^2 N$ (Theorems 4.10 and 5.12).

(c) $E(T) = \beta E(N) = 9(6.6) = 59.4$.

(d) Theorem 5.15 again:

$$E[V(T \mid N)] = \beta^2 E(N) = 81(6.6) = 534.6, \qquad V[E(T \mid N)] = \beta^2 V(N) = 81(45.04) = 3648.24,$$

$$V(T) = 534.6 + 3648.24 = 4182.84,$$

a standard deviation of 64.6749 thousand AZN. The weather risk $V(\lambda)$ enters $V(T)$ multiplied by β^2. Pooling more farms in the same district does not diversify it away, because every farm shares the same season.

Problem 20

seed 20260925

A development bank lends to small and medium-sized enterprises through regional branches. At a branch with 18 SME loans, the number Y of defaults in a year is binomial with parameters $n = 18$ and p, given that branch's default probability p. Across branches, p has a beta distribution with $\alpha = 1$ and $\beta = 38$. A branch is chosen at random.

Give every answer to at least four significant figures, as a decimal number.

(a) $E(p) =$ ____________

(b) $E(Y) =$ ____________

(c) $V(Y) =$ ____________

(d) The probability that the branch has no defaults at all, $P(Y = 0) =$ ____________

(Hint: condition on p and use the beta density. The answer is a product of 18 simple fractions.)

By Theorem 4.11, $E(p) = \alpha/(\alpha+\beta)$ and $V(p) = \alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$.

(a) $E(p) = 1/39 = 0.025641$.

(b) Given p, $E(Y \mid p) = np$, so by Theorem 5.14 $E(Y) = nE(p) = 18(0.025641) = 0.4615$.

(c) $V(p) = \frac{1(38)}{39^2(40)} = 0.00062459$ and $E(p^2) = V(p) + [E(p)]^2 = 0.00128205$. With $V(Y \mid p) = np(1-p)$, Theorem 5.15 gives

$$E[V(Y \mid p)] = n[E(p) - E(p^2)] = 18(0.025641 - 0.00128205) = 0.438462,$$

$$V[E(Y \mid p)] = n^2 V(p) = 324(0.00062459) = 0.202367,$$

$$V(Y) = 0.438462 + 0.202367 = 0.640828.$$

This equals $n\alpha\beta(\alpha+\beta+n)/[(\alpha+\beta)^2(\alpha+\beta+1)]$.

(d) $P(Y = 0) = E[I(Y = 0)] = E[P(Y = 0 \mid p)] = E[(1-p)^n]$, by Theorem 5.14 applied to the indicator of $\{Y = 0\}$. Against the beta density,

$$E[(1-p)^n] = \int_0^1 (1-p)^n \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}\,dp = \frac{B(\alpha,\beta+n)}{B(\alpha,\beta)} = \frac{\Gamma(\beta+n)\,\Gamma(\alpha+\beta)}{\Gamma(\beta)\,\Gamma(\alpha+\beta+n)}.$$

Since $\Gamma(x+1) = x\Gamma(x)$, the gamma functions telescope to a product of 18 factors:

$$P(Y=0) = \prod_{j=0}^{17} \frac{38+j}{39+j} = \frac{38}{39} \cdot \frac{38+1}{39+1} \cdots \frac{55}{56} = 0.678571.$$

For comparison, a binomial with p fixed at $E(p)$ gives $(1-0.025641)^{18} = 0.62653$, which is smaller. By Jensen's inequality $E[(1-p)^n] \geq (1-E(p))^n$. Branches with a low default rate are more likely to have a clean year, and the mixture keeps that.

Answer key

Problem	Answers
1	(a) 0, (b) 5.5, (c) 26.4, (d) 0.0178644
2	(a) 0.426471, (b) 0.4375, (c) 0.43
3	(a) y_2/2, (b) 24.3, (c) 18
4	(a) 0.314, (b) 0.738788, (c) 1727
5	(a) 4600, (b) 0.57, (c) 1311
6	(a) 17*p, (b) 0.085, (c) 1.445
7	(a) Model I only, (b) 0.00921863, (c) -0.3
8	(a) 2.24, (b) 3.6864, (c) 0.0617801
9	(a) 8.1, (b) 5.6, (c) 36, (d) 0.965994
10	(a) 730+0.518*(y_1-1000), (b) 0.518, (c) 1126.27, (d) 211.871
11	(a) 3*y_2^2/54872, (b) 2*y_2/3, (c) 12.6667, (d) 19
12	(a) y_1/2, (b) 23.25, (c) 193.028, (d) 231.549
13	(a) 4.34, (b) 4.4296, (c) 3.6673
14	(a) 1.52, (b) 2.6752, (c) 1.76, (d) 0.107008
15	(a) 159.6, (b) 175, (c) 909.72, (d) 1084.72
16	(a) 8.275, (b) 119.64, (c) 34.2019, (d) 153.842, (e) 22.2318
17	(a) 390.4, (b) 163.6, (c) 115, (d) No, ... are dependent, (e) 13.5+0.702934*[u-(-5.5)]
18	(a) 1.4, (b) 1.1, (c) 0.5, (d) 0.953, (e) 0.6669, (f) 0.117891, (g) 0.784791
19	(a) 6.6, (b) 45.04, (c) 59.4, (d) 4182.84
20	(a) 0.025641, (b) 0.461538, (c) 0.640828, (d) 0.678571