Week 15, Problem Set 2

Wackerly 5.12 - Chapter 5 in Review and a Comprehensive Review of Chapters 1-5

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A retail bank in Baku picks a customer at random from its books. Let A be the event that the customer holds one of the bank's credit cards and B the event that she holds a term deposit. From the customer database,

$$P(A) = 0.35, \qquad P(B) = 0.67, \qquad P(A \cap B) = 0.1545.$$

Give every probability to at least four significant figures.

(a) The probability that the customer holds at least one of the two products: $P(A \cup B) =$ ______________

(b) Among term-deposit holders, the proportion who also hold a card: $P(A \mid B) =$ ______________

(c) The probability that she holds neither product: ______________

(d) The probability that she holds exactly one of the two products: ______________

(e) Are the events A and B independent?

- ○ Independent
- ○ Dependent

Problem 2

seed 20260925

The monthly cargo throughput Y of a Caspian ferry terminal, in thousand tonnes, is modelled as a normal random variable with mean $\mu = 405$ and standard deviation $\sigma = 48$.

Use Table 4, Appendix 3, and give each probability to four decimal places.

(a) A month is called **busy** if throughput exceeds 458.76 thousand tonnes. $P(Y > 458.76) =$ ______________

(b) $P(328.20 < Y < 451.08) =$ ______________

(c) The terminal wants to plan storage for a throughput level that is exceeded in only 0.01 of months. Find the level y_0 with $P(Y > y_0) = 0.01$: $y_0 =$ ______________ thousand tonnes

(d) Throughputs in different months are independent. Find the probability that at least one of the next 6 months is busy (in the sense of part (a)): ______________

Problem 3

seed 20260925

A motor insurer models the size Y of a single damage claim, in thousand AZN, as exponential with mean $\beta = 2.4$:

$$f(y) = \frac{1}{2.4}\, e^{-y/2.4}, \qquad y > 0.$$

Each policy carries a deductible of 1.9 thousand AZN: the policyholder pays the first 1.9 thousand of any claim and the insurer pays the rest, $Y - 1.9$, if $Y > 1.9$ (and nothing otherwise).

Give answers to at least four significant figures.

(a) The probability that a claim exceeds the deductible: $P(Y > 1.9) =$ ______________

(b) Given that a claim exceeds the deductible, the probability that it exceeds 3.9 thousand AZN: $P(Y > 3.9 \mid Y > 1.9) =$ ______________

(c) The median claim size $\phi_{0.5}$ = ______________ thousand AZN

(d) The insurer's expected payment on a claim, including the claims on which it pays nothing: ______________ thousand AZN

Problem 4

seed 20260925

A mobile operator follows its corporate clients month by month. For a randomly chosen client-month let Y_1 be the number of service outages the client suffers (0, 1 or 2) and Y_2 the number of support tickets the client files (0, 1 or 2). The joint probability function is

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.1200	0.1400	0.1400
$y_1 = 1$	0.0550	0.1400	0.1050
$y_1 = 2$	0.0750	0.1200	0.1050

Give answers to at least four significant figures.

(a) $p_1(1) = P(Y_1 = 1) =$ ______________

(b) $P(Y_1 = 0 \mid Y_2 = 0) =$ ______________

(c) The expected number of tickets in a month with two outages: $E(Y_2 \mid Y_1 = 2) =$ ______________

(d) The probability that the client has more outages than tickets, $P(Y_1 > Y_2) =$ ______________

(e) Are Y_1 and Y_2 independent?

- ◯ Independent
- ◯ Not independent

Problem 5

seed 20260925

In each of its regional branches a bank holds two large SME loans and two large consumer loans. For a randomly chosen branch, let Y_1 be the number of its two SME loans that default this quarter and Y_2 the number of its two consumer loans that default (each 0, 1 or 2). The joint probability function is

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.64	0.06	0.02
$y_1 = 1$	0.07	0.08	0.02
$y_1 = 2$	0.02	0.04	0.05

Each SME default costs the bank 0.9 million AZN and each consumer default 0.3 million AZN, so the quarter's credit loss is $L = 0.9Y_1 + 0.3Y_2$ (million AZN).

Give answers to at least four significant figures.

(a) $E(Y_1) =$ ______________

(b) $E(Y_2) =$ ______________

(c) $E(Y_1Y_2) =$ ______________

(d) $\mathrm{Cov}(Y_1, Y_2) =$ ______________ (to four decimal places)

(e) $V(L) =$ ______________

Problem 6

seed 20260925

A pension fund holds shares of three energy companies: an oil producer (1), a renewables developer (2) and a regulated gas distributor (3). Annual returns R_1, R_2, R_3 (in percent) have

asset i	weight w_i	$E(R_i)$	σ_i
1	0.25	8.5	28
2	0.45	15.0	36
3	0.30	5.0	15

with correlation coefficients $\rho_{12} = 0.30$, $\rho_{13} = 0.10$ and $\rho_{23} = 0.35$. The portfolio return is $R_p = 0.25R_1 + 0.45R_2 + 0.30R_3$.

Give answers to at least four significant figures.

(a) $E(R_p) =$ ______________ %

(b) $\mathrm{Cov}(R_1, R_2) =$ ______________ (%^{2})

(c) $V(R_p) =$ ______________ (%^{2})

(d) The portfolio's standard deviation $\sigma_p =$ ______________ %

(e) $\mathrm{Cov}(R_p, R_1) =$ ______________ ($\%^2$)

Problem 7

seed 20260925

A motor insurer sorts its drivers into three risk classes. The share of drivers in each class and the probability that a driver of that class files at least one claim in a given year are

class	share of drivers	$P(A \mid B_i)$, claim this year
low (B_1)	0.53	0.05
medium (B_2)	0.29	0.17
high (B_3)	0.18	0.31

For a driver of a **given** class, claim years are independent of one another. The insurer cannot observe a new customer's class.

Give answers to at least four significant figures.

(a) The probability that a randomly chosen driver files a claim this year: ______________

(b) Given that a randomly chosen driver files a claim this year, the probability that the driver is high-risk:

____

(c) For a high-risk driver, let Y be the number of claim years among the next 5. Find $P(Y = 3 \mid B_3) =$

____

(d) The probability that a randomly chosen driver files claims in exactly 3 of the next 5 years: ______________

(e) A driver files claims in exactly 3 of 5 years. The probability that the driver is high-risk: ______________

Problem 8

seed 20260925

The number Y of service outages per month at a regional telephone exchange has a Poisson distribution with mean $\lambda = 0.8$. The operator's monthly cost for the exchange is a fixed monitoring cost of 6000 AZN plus 1400 AZN of compensation to business clients per outage: $C = 6000 + 1400Y$. Outage counts in different months are independent.

Give answers to at least four significant figures.

(a) The probability of an outage-free month, $P(Y = 0) =$ ______________

(b) $P(Y \geq 2) =$ ______________

(c) $E(C) =$ ______________ AZN

(d) The standard deviation of C: ______________ AZN

(e) Starting next month, the expected number of months up to and including the first outage-free month:

____

(f) The probability that the first outage-free month is month number 4: ______________

Problem 9

seed 20260925

After a fault on a distribution feeder, the time Y (hours) until power is restored has a gamma distribution with $\alpha = 2$ and $\beta = 1.9$, i.e.

$$f(y) = \frac{y\,e^{-y/1.9}}{(1.9)^2}, \qquad y > 0.$$

For each fault the utility pays a fixed call-out cost of 850 AZN plus a penalty to the regulator that grows with the square of the outage length, $C = 850 + 120Y^2$ (AZN).

Give answers to at least four significant figures.

(a) The mean restoration time $E(Y) =$ ______________ hours

(b) The standard deviation of Y: ______________ hours

(c) The probability that restoration takes more than 3.0 hours: $P(Y > 3.0) =$ ______________

(d) The expected cost of a fault, $E(C) =$ ______________ AZN

Problem 10

seed 20260925

A trade-credit insurer covers exporters against buyers who fail to pay. When a buyer defaults, the proportion Y of the invoice eventually recovered through collection has density

$$f(y) = \begin{cases} k\,y^1(1-y)^3, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

a beta density with $\alpha = 2$ and $\beta = 4$.

Give answers to at least four significant figures.

(a) The constant $k =$ ______________

(b) The mean recovery rate $E(Y) =$ ______________

(c) $V(Y) =$ ______________

(d) The probability that more than 0.45 of the invoice is recovered, $P(Y > 0.45) =$ ______________

(e) A defaulted invoice is for 360 thousand AZN. The insurer pays the unrecovered part, $360(1 - Y)$. Its expected payment: ______________ thousand AZN

Problem 11

seed 20260925

A container sent by rail from Baku to the Georgian border takes at least 4 days; beyond that the delay is exponential with mean 2.8 days. The transit time Y (days) therefore has density

$$f(y) = \begin{cases} \frac{1}{2.8}\, e^{-(y-4)/2.8}, & y \geq 4, \\ 0, & y < 4. \end{cases}$$

(a) Using Definition 4.14, find the moment-generating function of Y (valid for $t < 1/2.8$). Enter it as a function of t, e.g. e^(2*t)/(1-0.5*t).

$m_Y(t) =$ ______________

(b) Use Theorem 3.12 on your $m_Y(t)$ to find $E(Y) =$ ______________ days

(c) $V(Y) =$ ______________ days^2

(d) The freight forwarder charges $U = 4 + 1.4Y$ (hundred AZN). Find the moment-generating function of U:

$m_U(t) =$ ______________

Problem 12

seed 20260925

A property insurer records every claim above 190 thousand AZN. The size Y (thousand AZN) of a recorded claim has the Pareto distribution function

$$F(y) = \begin{cases} 0, & y < 190, \\ 1 - \left(\frac{190}{y}\right)^{2.5}, & y \geq 190. \end{cases}$$

Give answers to at least four significant figures.

(a) The density function for $y > 190$, as a function of y: $f(y) =$ ______________

(b) The 0.86 quantile $\phi_{0.86}$ of claim size: ______________ thousand AZN

(c) Given that a claim exceeds 570 thousand AZN, the probability that it exceeds 950 thousand AZN:

____

(d) $E(Y) =$ ______________ thousand AZN

(e) A reinsurer pays the part of each claim above a retention of 950 thousand AZN, that is $Y - 950$ if $Y > 950$ and 0 otherwise. Its expected payment per recorded claim: ______________ thousand AZN

Problem 13

seed 20260925

The total cash Y (thousand AZN) withdrawn in a day from a busy ATM in central Baku has mean $\mu = 28$ and standard deviation $\sigma = 28$. Take $k = 2.88$.

Give answers to at least four significant figures, except (c), which should be given to four decimal places using Table 4.

(a) Without assuming any particular distribution, the largest lower bound that Tchebysheff's theorem gives for $P(|Y - \mu| < k\sigma)$: ______________

(b) The exact value of $P(|Y-\mu|<k\sigma)$ if Y is exponential with mean 28: ______________

(c) The exact value of $P(|Y-\mu|<k\sigma)$ if instead Y were normal with the same mean and standard deviation:

____

(d) The bank loads the ATM each morning with c thousand AZN and wants $P(Y>c)=0.05$. Assuming Y is exponential with mean 28, $c=$ ______________

(e) A risk officer who does not trust the exponential model instead loads $c=\mu+k^*\sigma$, with k^* chosen so that Tchebysheff's theorem guarantees $P(|Y-\mu|\geq k^*\sigma)\leq 0.05$ (and hence $P(Y>c)\leq 0.05$). Find this $c=$

____

Problem 14

seed 20260925

A telecom operator leases backbone capacity to an internet provider. For a randomly chosen month let Y_1 be the proportion of the link the provider reserves and Y_2 the proportion it actually uses; usage cannot exceed the reservation. Their joint density is

$$f(y_1,y_2)=\begin{cases} k\,y_1^1, & 0\leq y_2\leq y_1\leq 1,\\ 0, & \text{elsewhere.}\end{cases}$$

Give answers to at least four significant figures.

(a) The constant $k=$ ______________

(b) $P(Y_1\leq 0.85)=$ ______________

(c) $P(Y_2\leq 0.30)=$ ______________

(d) In a month in which the provider reserves 0.85 of the link, the probability that it uses at most 0.25 of it: $P(Y_2\leq 0.25\mid Y_1=0.85)=$ ______________

(e) The expected proportion used, $E(Y_2)=$ ______________

Problem 15

seed 20260925

An insurer runs a motor line and a property line with separate customers and separate call centres. Let Y_1 be the delay (days) before a motor accident is reported and Y_2 the delay before a property loss is reported, for one claim drawn at random from each line. The actuaries fit the joint density

$$f(y_1,y_2)=\begin{cases} \dfrac{1}{18.75}\,e^{-y_1/2.5-y_2/7.5}, & y_1>0,\ y_2>0,\\ 0, & \text{elsewhere.}\end{cases}$$

Give answers to at least four significant figures.

(a) $P(Y_1>2,\ Y_2>8)=$ ______________

(b) The probability that the motor claim is reported first, $P(Y_1<Y_2)=$ ______________

(c) The probability that the two delays add up to at most 8 days, $P(Y_1+Y_2\leq 8)=$ ______________

(d) $E(Y_1Y_2) =$ ______________

(e) $V(Y_1 - Y_2) =$ ______________

Problem 16

seed 20260925

At a Caspian port, each import declaration is routed independently to the **green** channel (released without checks) with probability 0.73, the **yellow** channel (documentary check) with probability 0.19, or the **red** channel (physical inspection) with probability 0.08. A logistics firm files 9 declarations this week. Let Y_1, Y_2, Y_3 be the numbers routed green, yellow and red.

A documentary check costs the firm 85 AZN and a physical inspection 350 AZN, so its inspection cost is $C = 85Y_2 + 350Y_3$.

Give answers to at least four significant figures (enter decimals).

(a) $P(Y_1 = 4, Y_2 = 4, Y_3 = 1) =$ ______________

(b) The probability that none of the 9 declarations is sent to the red channel: ______________

(c) $\mathrm{Cov}(Y_2, Y_3) =$ ______________

(d) $E(C) =$ ______________ AZN

(e) $V(C) =$ ______________ AZN^2

Problem 17

seed 20260925

A household-insurance portfolio is not homogeneous: each policyholder has an unobserved annual claim rate Λ, which varies across policyholders as a gamma random variable with $\alpha = 2.0$ and $\beta = 0.28$. Given $\Lambda = \lambda$, the number N of claims the policyholder makes in a year is Poisson with mean λ. Claim sizes $X_1, X_2, \ldots$ are independent of each other and of N, each with mean 1600 AZN and standard deviation 1900 AZN. The policyholder's aggregate claim for the year is $T = X_1 + \cdots + X_N$ ($T = 0$ if $N = 0$).

Give answers to at least four significant figures.

(a) For a randomly chosen policyholder, $E(N) =$ ______________

(b) $V(N) =$ ______________

(c) $P(N = 0) =$ ______________

(d) $E(T) =$ ______________ AZN

(e) The standard deviation of T: ______________ AZN

Problem 18

seed 20260925

A gas utility injects gas into an underground storage site over the summer and withdraws it during the winter. The amount injected, Y_1 (million cubic metres), has a gamma distribution with $\alpha = 2$ and $\beta = 7.5$. Given $Y_1 = y_1$, the amount withdrawn over the winter, Y_2, is uniformly distributed on $(0, y_1)$ (how much is needed depends on how cold the winter is).

Give answers to at least four significant figures.

(a) In a year when 15.0 million cubic metres were injected, the probability that more than 12.0 are withdrawn: $P(Y_2 > 12.0 \mid Y_1 = 15.0) =$ ______________

(b) $E(Y_2) =$ ______________

(c) $V(Y_2) =$ ______________

(d) $\mathrm{Cov}(Y_1, Y_2) =$ ______________

(e) The coefficient of correlation ρ between Y_1 and Y_2: ______________

Problem 19

seed 20260925

A grid operator monitors a high-voltage line into Baku. Measured in months from now, let Y_1 be the time of the line's first protective trip and Y_2 the time of its second. Their joint density is

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{20.25}\, e^{-y_2/4.5}, & 0 < y_1 < y_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures.

(a) The probability that there is no trip in the next 6.0 months, $P(Y_1 > 6.0) =$ ______________

(b) The probability that the line trips at least twice within 13 months, $P(Y_2 \leq 13) =$ ______________

(c) $P(Y_1 \leq 4 \mid Y_2 = 19) =$ ______________

(d) $\mathrm{Cov}(Y_1, Y_2) =$ ______________

(e) $V(Y_1 + Y_2) =$ ______________

Problem 20

seed 20260925

Let Y_1 be next month's return (in %) on a regional equity index and Y_2 the return on the shares of a bank listed on it. The pair has a bivariate normal distribution with

$$\mu_1 = 0.7, \quad \sigma_1 = 8, \qquad \mu_2 = 2.30, \quad \sigma_2 = 10, \qquad \rho = 0.6.$$

Recall (Exercise 5.129) that, given $Y_1 = y_1$, Y_2 is normal with mean $\mu_2 + \rho\dfrac{\sigma_2}{\sigma_1}(y_1 - \mu_1)$ and variance $\sigma_2^2(1 - \rho^2)$.

Give (b) and (e) to four decimal places using Table 4; give the others to at least four significant figures.

(a) $\mathrm{Cov}(Y_1, Y_2) =$ ______________

(b) The probability that the bank's shares lose value next month, $P(Y_2 < 0) =$ ______________

(c) A stress scenario assumes the index returns $y_1 = -14.34$. Find $E(Y_2 \mid Y_1 = -14.34) =$ ____________

(d) The conditional standard deviation of Y_2 given $Y_1 = -14.34$: ____________

(e) In the stress scenario, the probability that the bank's shares fall below -18.82 %:
$P(Y_2 < -18.82 \mid Y_1 = -14.34) =$ ____________