Week 15, Problem Set 2 - worked solutions

Wackerly 5.12 - Chapter 5 in Review and a Comprehensive Review of Chapters 1-5

> This is one randomly generated version of the set. Every student receives different numbers, so the values here will not match your own copy — follow the method, not the arithmetic.

Problem 1

seed 20260925

A retail bank in Baku picks a customer at random from its books. Let A be the event that the customer holds one of the bank's credit cards and B the event that she holds a term deposit. From the customer database,

$$P(A) = 0.35, \qquad P(B) = 0.67, \qquad P(A \cap B) = 0.1545.$$

Give every probability to at least four significant figures.

(a) The probability that the customer holds at least one of the two products: $P(A \cup B) =$ ______________

(b) Among term-deposit holders, the proportion who also hold a card: $P(A \mid B) =$ ______________

(c) The probability that she holds neither product: ______________

(d) The probability that she holds exactly one of the two products: ______________

(e) Are the events A and B independent?

- ◯ Independent
- ◯ Dependent

(a) The additive law (Theorem 2.6) removes the double-counted intersection:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.35 + 0.67 - 0.1545 = 0.8655.$$

(b) Definition 2.9:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1545}{0.67} = 0.2306.$$

(c) "Neither" is $\bar{A} \cap \bar{B} = \overline{A \cup B}$ (De Morgan), so by Theorem 2.7 its probability is $1 - P(A \cup B) = 1 - 0.8655 = 0.1345$.

(d) "Exactly one" is $A \cup B$ with the intersection removed. The intersection sits inside both A and B, so it must be subtracted twice from $P(A) + P(B)$:

$$P(\text{exactly one}) = P(A) + P(B) - 2P(A \cap B) = 0.35 + 0.67 - 0.3090 = 0.7110.$$

Check: $P(A \cup B) - P(A \cap B) = 0.8655 - 0.1545 = 0.7110$.

(e) Definition 2.10 asks whether $P(A \cap B) = P(A)P(B)$. Here $P(A)P(B) = 0.35 \times 0.67 = 0.2345$ against $P(A \cap B) = 0.1545$. They differ, so A and B are dependent. Here P(A | B) < P(A): deposit holders are less likely than average to hold a card.

Equivalently compare (b) with $P(A) = 0.35$: independence is exactly the statement $P(A \mid B) = P(A)$.

Problem 2

seed 20260925

The monthly cargo throughput Y of a Caspian ferry terminal, in thousand tonnes, is modelled as a normal random variable with mean $\mu = 405$ and standard deviation $\sigma = 48$.

Use Table 4, Appendix 3, and give each probability to four decimal places.

(a) A month is called **busy** if throughput exceeds 458.76 thousand tonnes. $P(Y > 458.76) =$ ______________

(b) $P(328.20 < Y < 451.08) =$ ______________

(c) The terminal wants to plan storage for a throughput level that is exceeded in only 0.01 of months. Find the level y_0 with $P(Y > y_0) = 0.01$: $y_0 =$ ______________ thousand tonnes

(d) Throughputs in different months are independent. Find the probability that at least one of the next 6 months is busy (in the sense of part (a)): ______________

Standardise with $Z = (Y - \mu)/\sigma = (Y - 405)/48$, which is standard normal. Table 4 gives the right-tail area $A(z) = P(Z > z)$ for $z \geq 0$.

(a) $z = (458.76 - 405)/48 = 1.12$, so $P(Y > 458.76) = P(Z > 1.12) = A(1.12) = 0.1314$.

(b) The endpoints standardise to -1.60 and 0.96. Removing both tails, and using symmetry for the left one,

$$P(-1.60 < Z < 0.96) = 1 - A(1.60) - A(0.96) = 1 - 0.0548 - 0.1685 = 0.7767.$$

(c) This is the upper 0.01 point: find z_0 with $A(z_0) = 0.01$ in the body of Table 4, which gives $z_0 = 2.33$. Undo the standardisation (the quantile of Definition 4.4, with $p = 1 - 0.01$):

$$y_0 = \mu + z_0\sigma = 405 + 2.33(48) = 516.84.$$

(d) Each month is busy with probability $p = 0.1314$, independently, so the number of busy months X among the next 6 is binomial with $n = 6$ and this p (Definition 3.7). Then

$$P(X \geq 1) = 1 - P(X = 0) = 1 - (1 - p)^6 = 1 - 0.8686^6 = 0.5704.$$

A continuous model (the normal) decides what happens in one month; a discrete model (the binomial) counts how often it happens across months. Exam problems often chain the two in exactly this way.

Problem 3

seed 20260925

A motor insurer models the size Y of a single damage claim, in thousand AZN, as exponential with mean $\beta = 2.4$:

$$f(y) = \frac{1}{2.4}\, e^{-y/2.4}, \qquad y > 0.$$

Each policy carries a deductible of 1.9 thousand AZN: the policyholder pays the first 1.9 thousand of any claim and the insurer pays the rest, $Y - 1.9$, if $Y > 1.9$ (and nothing otherwise).

Give answers to at least four significant figures.

(a) The probability that a claim exceeds the deductible: $P(Y > 1.9) =$ ______________

(b) Given that a claim exceeds the deductible, the probability that it exceeds 3.9 thousand AZN: $P(Y > 3.9 \mid Y > 1.9) =$

____

(c) The median claim size $\phi_{0.5}$ = ______________ thousand AZN

(d) The insurer's expected payment on a claim, including the claims on which it pays nothing: ______________ thousand AZN

The distribution function is $F(y) = 1 - e^{-y/2.4}$ for $y > 0$, so $P(Y > y) = e^{-y/2.4}$.

(a) $P(Y > 1.9) = e^{-1.9/2.4} = e^{-0.7917} = 0.4531$.

(b) By Definition 2.9, and because $\{Y > 3.9\} \subset \{Y > 1.9\}$,

$$P(Y > 3.9 \mid Y > 1.9) = \frac{P(Y > 3.9)}{P(Y > 1.9)} = \frac{e^{-3.9/2.4}}{e^{-1.9/2.4}} = e^{-2.0/2.4} = 0.4346.$$

This equals $P(Y > 2.0)$: the exponential is **memoryless** (Section 4.6). Having passed the deductible, the excess behaves like a brand-new claim.

(c) Definition 4.4: $F(\phi_{0.5}) = 0.5$, i.e. $e^{-\phi/2.4} = 0.5$, so $\phi_{0.5} = 2.4 \ln 2 = 1.6636$. The median is below the mean $\beta = 2.4$, as it must be for a right-skewed density.

(d) The payment is $g(Y) = Y - 1.9$ for $Y > 1.9$ and 0 otherwise. By Theorem 4.4, with the substitution $u = y - 1.9$,

$$E[g(Y)] = \int_{1.9}^{\infty} (y - 1.9) \frac{1}{2.4} e^{-y/2.4}\, dy = e^{-1.9/2.4} \int_{0}^{\infty} u \frac{1}{2.4} e^{-u/2.4}\, du = e^{-1.9/2.4}\, 2.4 = 1.0874.$$

The last integral is the mean of an exponential with mean 2.4 (Theorem 4.10). The same answer follows from (a) and (b): the insurer pays with probability 0.4531, and, by memorylessness, when it pays the expected excess is the full mean 2.4.

Problem 4

seed 20260925

A mobile operator follows its corporate clients month by month. For a randomly chosen client-month let Y_1 be the number of service outages the client suffers (0, 1 or 2) and Y_2 the number of support tickets the client files (0, 1 or 2). The joint probability function is

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.1200	0.1400	0.1400
$y_1 = 1$	0.0550	0.1400	0.1050
$y_1 = 2$	0.0750	0.1200	0.1050

Give answers to at least four significant figures.

(a) $p_1(1) = P(Y_1 = 1) =$ ____________

(b) $P(Y_1 = 0 \mid Y_2 = 0) =$ ____________

(c) The expected number of tickets in a month with two outages: $E(Y_2 \mid Y_1 = 2) =$ ____________

(d) The probability that the client has more outages than tickets, $P(Y_1 > Y_2) =$ ____________

(e) Are Y_1 and Y_2 independent?

- ○ Independent
- ○ Not independent

(a) Definition 5.4: sum the row $y_1 = 1$, $p_1(1) = 0.0550 + 0.1400 + 0.1050 = 0.3000$.

(b) Definition 5.5 needs $p_2(0) = 0.1200 + 0.0550 + 0.0750 = 0.2500$. Then

$$P(Y_1 = 0 \mid Y_2 = 0) = \frac{p(0,0)}{p_2(0)} = \frac{0.1200}{0.2500} = 0.4800.$$

(c) First $p_1(2) = 0.0750 + 0.1200 + 0.1050 = 0.3000$, and the conditional probability function is $p(y_2 \mid 2) = p(2, y_2)/p_1(2)$. Definition 5.13:

$$E(Y_2 \mid Y_1 = 2) = \sum_{y_2} y_2 \frac{p(2, y_2)}{p_1(2)} = \frac{0(0.0750) + 1(0.1200) + 2(0.1050)}{0.3000} = \frac{0.1200 + 0.2100}{0.3000} = 1.1000.$$

(d) The cells with $y_1 > y_2$ are $(1, 0), (2, 0), (2, 1)$: $P(Y_1 > Y_2) = 0.0550 + 0.0750 + 0.1200 = 0.2500$.

(e) Theorem 5.4: independence holds if and only if $p(y_1, y_2) = p_1(y_1) p_2(y_2)$ for **every** pair. The (0,0) cell already fails: p(0,0) = 0.1200 but p1(0) p2(0) = 0.4000 x 0.2500 = 0.1000. One failing cell is enough, so by Theorem 5.4 the counts are not independent.

Problem 5

seed 20260925

In each of its regional branches a bank holds two large SME loans and two large consumer loans. For a randomly chosen branch, let Y_1 be the number of its two SME loans that default this quarter and Y_2 the number of its two consumer loans that default (each 0, 1 or 2). The joint probability function is

$p(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$
$y_1 = 0$	0.64	0.06	0.02
$y_1 = 1$	0.07	0.08	0.02
$y_1 = 2$	0.02	0.04	0.05

Each SME default costs the bank 0.9 million AZN and each consumer default 0.3 million AZN, so the quarter's credit loss is $L = 0.9Y_1 + 0.3Y_2$ (million AZN).

Give answers to at least four significant figures.

(a) $E(Y_1) =$ ______________

(b) $E(Y_2) =$ ______________

(c) $E(Y_1Y_2) =$ ______________

(d) $\mathrm{Cov}(Y_1, Y_2) =$ ______________ (to four decimal places)

(e) $V(L) =$ ______________

Row totals (Definition 5.4): $p_1(0), p_1(1), p_1(2) = 0.72, 0.17, 0.11$. Column totals: $p_2(0), p_2(1), p_2(2) = 0.73, 0.18, 0.09$.

(a) $E(Y_1) = 0(0.72) + 1(0.17) + 2(0.11) = 0.39$.

(b) $E(Y_2) = 0(0.73) + 1(0.18) + 2(0.09) = 0.36$.

(c) Definition 5.9 with $g = y_1y_2$; only cells with both counts positive contribute:

$$E(Y_1Y_2) = 1 \cdot 1(0.08) + 1 \cdot 2(0.02) + 2 \cdot 1(0.04) + 2 \cdot 2(0.05) = 0.40.$$

(d) Theorem 5.10: $\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0.40 - 0.39(0.36) = 0.2596$. The covariance is positive: the two segments tend to go bad together, which adds to the risk of the total loss.

(e) The variances first (Theorem 3.6): $E(Y_1^2) = 0.17 + 4(0.11) = 0.61$, so $V(Y_1) = 0.61 - 0.39^2 = 0.4579$; and $E(Y_2^2) = 0.18 + 4(0.09) = 0.54$, so $V(Y_2) = 0.54 - 0.36^2 = 0.4104$. Theorem 5.12 for $L = aY_1 + bY_2$:

$$V(L) = a^2V(Y_1) + b^2V(Y_2) + 2ab\,\mathrm{Cov}(Y_1, Y_2) = 0.81(0.4579) + 0.09(0.4104) + 0.54(0.2596) = 0.5480.$$

The standard deviation of the quarterly loss is $\sqrt{0.5480} = 0.7403$ million AZN.

Problem 6

seed 20260925

A pension fund holds shares of three energy companies: an oil producer (1), a renewables developer (2) and a regulated gas distributor (3). Annual returns R_1, R_2, R_3 (in percent) have

asset i	weight w_i	$E(R_i)$	σ_i
1	0.25	8.5	28
2	0.45	15.0	36
3	0.30	5.0	15

with correlation coefficients $\rho_{12} = 0.30$, $\rho_{13} = 0.10$ and $\rho_{23} = 0.35$. The portfolio return is $R_p = 0.25R_1 + 0.45R_2 + 0.30R_3$.

Give answers to at least four significant figures.

(a) $E(R_p) =$ ______________ %

(b) $\text{Cov}(R_1, R_2) =$ ______________ (%^{2})

(c) $V(R_p) =$ ______________ (%^{2})

(d) The portfolio's standard deviation $\sigma_p =$ ______________ %

(e) $\text{Cov}(R_p, R_1) =$ ______________ (%^{2})

(a) Theorem 5.12, part 1: $E(R_p) = \sum w_i E(R_i) = 0.25(8.5) + 0.45(15.0) + 0.30(5.0) = 10.3750$ %.

(b) The correlation coefficient is $\rho_{12} = \text{Cov}(R_1, R_2)/(\sigma_1\sigma_2)$, so $\text{Cov}(R_1, R_2) = 0.30(28)(36) = 302.400$. Likewise $\text{Cov}(R_1, R_3) = 0.10(28)(15) = 42.000$ and $\text{Cov}(R_2, R_3) = 0.35(36)(15) = 189.000$.

(c) Theorem 5.12, part 2: \[V(R_p) = \sum_i w_i^2\sigma_i^2 + 2\sum_{i

(d) $\sigma_p = \sqrt{457.0600} = 21.3790$ %. Because every correlation is below 1, this is less than the weighted average of the three standard deviations: diversification at work.

(e) Theorem 5.12, part 3, with $U_1 = R_p$ and $U_2 = R_1$:

$\text{Cov}(R_p, R_1) = 0.25V(R_1) + 0.45\text{Cov}(R_2, R_1) + 0.30\text{Cov}(R_3, R_1) = 0.25(784) + 0.45(302.400) + 0.30(42.000) = 344.6800.$

The implied correlation between the fund and the oil share is $344.6800/(21.3790 \times 28) = 0.5758$.

Problem 7

seed 20260925

A motor insurer sorts its drivers into three risk classes. The share of drivers in each class and the probability that a driver of that class files at least one claim in a given year are

class	share of drivers	$P(A \mid B_i)$, claim this year
low (B_1)	0.53	0.05
medium (B_2)	0.29	0.17
high (B_3)	0.18	0.31

For a driver of a **given** class, claim years are independent of one another. The insurer cannot observe a new customer's class.

Give answers to at least four significant figures.

(a) The probability that a randomly chosen driver files a claim this year: ______________

(b) Given that a randomly chosen driver files a claim this year, the probability that the driver is high-risk: ______________

(c) For a high-risk driver, let Y be the number of claim years among the next 5. Find $P(Y = 3 \mid B_3) =$ ______________

(d) The probability that a randomly chosen driver files claims in exactly 3 of the next 5 years: ______________

(e) A driver files claims in exactly 3 of 5 years. The probability that the driver is high-risk: ______________

The classes B_1, B_2, B_3 form a partition (Definition 2.11); let A be "a claim this year".

(a) Theorem 2.8:

$$P(A) = \sum_i P(A \mid B_i)P(B_i) = 0.05(0.53) + 0.17(0.29) + 0.31(0.18) = 0.1316.$$

(b) Bayes' rule, Theorem 2.9:

$$P(B_3 \mid A) = \frac{P(A \mid B_3)P(B_3)}{P(A)} = \frac{0.0558}{0.1316} = 0.4240.$$

One claim raises the probability of "high risk" from the prior 0.18 to 0.4240.

(c) Given B_3, the 5 years are independent trials, each a "claim year" with probability 0.31, so Y is binomial (Definition 3.7):

$$P(Y=3 \mid B_3) = \binom{5}{3}(0.31)^3(0.69)^2 = 10(0.31)^3(0.69)^2 = 0.141835.$$

The same formula with 0.05 and 0.17 gives $P(Y=3 \mid B_1) = 0.001128$ and $P(Y=3 \mid B_2) = 0.033846$.

(d) Theorem 2.8 again, now with the binomial probabilities as the conditional probabilities:

$$P(Y=3) = 0.001128(0.53) + 0.033846(0.29) + 0.141835(0.18) = 0.035943.$$

(e) Theorem 2.9:

$$P(B_3 \mid Y=3) = \frac{0.141835(0.18)}{0.035943} = \frac{0.025530}{0.035943} = 0.7103.$$

This is experience rating: the claim history is evidence about the unobserved class, and Bayes' rule converts it into an updated risk assessment on which a premium can be based. Notice that the years are independent **within** a class but not for a randomly chosen driver: a claim this year makes "high risk" more likely, and so makes a claim next year more likely too.

Problem 8

seed 20260925

The number Y of service outages per month at a regional telephone exchange has a Poisson distribution with mean $\lambda = 0.8$. The operator's monthly cost for the exchange is a fixed monitoring cost of 6000 AZN plus 1400 AZN of compensation to business clients per outage: $C = 6000 + 1400Y$. Outage counts in different months are independent.

Give answers to at least four significant figures.

(a) The probability of an outage-free month, $P(Y=0) =$ ______________

(b) $P(Y \geq 2) =$ ______________

(c) $E(C) =$ ______________ AZN

(d) The standard deviation of C: ______________ AZN

(e) Starting next month, the expected number of months up to and including the first outage-free month: ______________

(f) The probability that the first outage-free month is month number 4: ______________

Definition 3.11 gives $p(y) = \lambda^y e^{-\lambda}/y!$ with $\lambda = 0.8$.

(a) $p(0) = e^{-0.8} = 0.4493$.

(b) By the complement rule,

$$P(Y \geq 2) = 1 - p(0) - p(1) = 1 - e^{-0.8} - 0.8e^{-0.8} = 1 - 0.4493 - 0.3595 = 0.1912.$$

(c) Theorem 3.11 gives $E(Y) = \lambda = 0.8$; Theorems 3.3-3.5 give $E(C) = 6000 + 1400E(Y) = 6000 + 1400(0.8) = 7120.0$ AZN.

(d) Theorem 3.11 also gives $V(Y) = \lambda$. Adding a constant does not change a variance and multiplying by 1400 multiplies it by 1400^2, so $V(C) = 1400^2(0.8)$ and $\sigma_C = 1400\sqrt{0.8} = 1252.20$ AZN.

(e) Each month is a trial that "succeeds" (no outage) with probability $p = 0.4493$, independently of the other months. The number of months up to and including the first success is geometric (Definition 3.8), and by Theorem 3.8 its mean is $1/p = 1/0.4493 = 2.2255$ months.

(f) Each of the first 3 months must contain an outage and month 4 must not:

$$p(4) = q^3 p = (0.5507)^3(0.4493) = 0.0750.$$

The two models answer different questions about the same process: the Poisson counts events within a month, the geometric counts months until a month without events.

Problem 9

seed 20260925

After a fault on a distribution feeder, the time Y (hours) until power is restored has a gamma distribution with $\alpha = 2$ and $\beta = 1.9$, i.e.

$$f(y) = \frac{y\,e^{-y/1.9}}{(1.9)^2}, \qquad y > 0.$$

For each fault the utility pays a fixed call-out cost of 850 AZN plus a penalty to the regulator that grows with the square of the outage length, $C = 850 + 120Y^2$ (AZN).

Give answers to at least four significant figures.

(a) The mean restoration time $E(Y) =$ ____________ hours

(b) The standard deviation of Y: ____________ hours

(c) The probability that restoration takes more than 3.0 hours: $P(Y > 3.0) =$ ____________

(d) The expected cost of a fault, $E(C) =$ ____________ AZN

Definition 4.9 with $\alpha = 2$ gives the density shown, since $\Gamma(2) = 1$.

(a) Theorem 4.8: $E(Y) = \alpha\beta = 2(1.9) = 3.80$ hours.

(b) Theorem 4.8: $V(Y) = \alpha\beta^2 = 2(1.9)^2 = 7.2200$, so $\sigma = \sqrt{7.2200} = 2.6870$ hours.

(c) $P(Y > c) = \int_c^\infty f(y)\,dy$. With $u = y/\beta$ this is $\int_{c/\beta}^\infty u^{\alpha-1}e^{-u}\,du/\Gamma(\alpha)$, and each integration by parts lowers the power of u by one. Integrating by parts once gives

$$P(Y > c) = e^{-c/\beta}\left(1 + \frac{c}{\beta}\right).$$

Here $c/\beta = 3.0/1.9 = 1.5789$, so

$$P(Y > 3.0) = e^{-1.5789}(1 + 1.5789) = 0.5318.$$

(The bracket is the probability that a Poisson variable with mean c/β is below α: the classic gamma-Poisson link.)

(d) Theorem 4.5 gives $E(C) = 850 + 120E(Y^2)$, and by Theorem 3.6's continuous analogue $E(Y^2) = V(Y) + [E(Y)]^2 = 7.2200 + 3.80^2 = 21.6600$ (equivalently $\alpha(\alpha+1)\beta^2 = 2(3)(1.9)^2$). Hence

$$E(C) = 850 + 120(21.6600) = 3449.20 \text{ AZN}.$$

Note that $E(C) \neq 850 + 120[E(Y)]^2$: the expected value of a nonlinear function is not the function of the expected value, and the difference, $120V(Y)$, is the price of variability.

Problem 10

seed 20260925

A trade-credit insurer covers exporters against buyers who fail to pay. When a buyer defaults, the proportion Y of the invoice eventually recovered through collection has density

$$f(y) = \begin{cases} k\,y^1(1-y)^3, & 0 \le y \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

a beta density with $\alpha = 2$ and $\beta = 4$.

Give answers to at least four significant figures.

(a) The constant $k =$ ______________

(b) The mean recovery rate $E(Y) =$ ______________

(c) $V(Y) =$ ______________

(d) The probability that more than 0.45 of the invoice is recovered, $P(Y > 0.45) =$ ______________

(e) A defaulted invoice is for 360 thousand AZN. The insurer pays the unrecovered part, $360(1 - Y)$. Its expected payment: ______________ thousand AZN

(a) Definition 4.12 gives $k = 1/B(\alpha, \beta) = \Gamma(\alpha + \beta)/(\Gamma(\alpha)\Gamma(\beta))$. With integer arguments $\Gamma(n) = (n-1)!$, so

$$k = \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)} = \frac{120}{1 \cdot 6} = 20.$$

(Check: $\int_0^1 f(y)\,dy = 1$ with this k.)

(b) Theorem 4.11: $E(Y) = \alpha/(\alpha + \beta) = 2/6 = 0.3333$.

(c) Theorem 4.11:

$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2(4)}{6^2(7)} = 0.031746.$$

(d) Expand the power of (1 - y) with the binomial theorem and integrate term by term from 0 to 0.45:

$$F(0.45) = k \sum_{j=0}^{3} \binom{3}{j} (-1)^j \frac{0.45^{2+j}}{2+j} = 0.7438,$$

so $P(Y > 0.45) = 1 - 0.7438 = 0.2562$. (For integer parameters this also equals the probability that a binomial variable with $n = \alpha + \beta - 1 = 5$ trials and success probability 0.45 is at most $\alpha - 1 = 1$, a useful check.)

(e) By Theorem 4.5, $E[360(1 - Y)] = 360(1 - E(Y)) = 360(1 - 0.3333) = 240.00$ thousand AZN. Only the mean of Y is needed for an expected loss; the variance in (c) is what the insurer needs to price the **uncertainty** around it.

Problem 11

seed 20260925

A container sent by rail from Baku to the Georgian border takes at least 4 days; beyond that the delay is exponential with mean 2.8 days. The transit time Y (days) therefore has density

$$f(y) = \begin{cases} \dfrac{1}{2.8} e^{-(y-4)/2.8}, & y \geq 4, \\ 0, & y < 4. \end{cases}$$

(a) Using Definition 4.14, find the moment-generating function of Y (valid for $t < 1/2.8$). Enter it as a function of t, e.g. e^(2*t)/(1-0.5*t).

$m_Y(t) =$ ______________

(b) Use Theorem 3.12 on your $m_Y(t)$ to find $E(Y) =$ ______________ days

(c) $V(Y) =$ ______________ days^2

(d) The freight forwarder charges $U = 4 + 1.4Y$ (hundred AZN). Find the moment-generating function of U:

$m_U(t) =$ ______________

(a) By Definition 4.14 and Theorem 4.4,

$$m_Y(t) = E(e^{tY}) = \int_4^\infty e^{ty} \frac{1}{2.8} e^{-(y-4)/2.8}\, dy.$$

Substitute $u = y - 4$, so $e^{ty} = e^{4t}e^{tu}$:

$$m_Y(t) = e^{4t} \int_0^\infty \frac{1}{2.8} e^{-u(1/2.8-t)}\, du = e^{4t} \cdot \frac{1/2.8}{1/2.8 - t} = \frac{e^{4t}}{1-2.8t},$$

valid when $1/2.8 - t > 0$, i.e. $t < 0.3571$. The integral is the exponential MGF $(1-\beta t)^{-1}$ multiplied by the MGF $e^{\theta t}$ of the constant shift.

(b) Write $m_Y(t) = e^{4t}(1-2.8t)^{-1}$. By the product rule,

$$m_Y'(t) = 4e^{4t}(1-2.8t)^{-1} + 2.8e^{4t}(1-2.8t)^{-2},$$

so by Theorem 3.12 $E(Y) = m_Y'(0) = 4 + 2.8 = 6.8$ days.

(c) Differentiating again and setting $t = 0$ gives $E(Y^2) = m_Y''(0) = \theta^2 + 2\theta\beta + 2\beta^2$, hence

$$V(Y) = E(Y^2) - [E(Y)]^2 = \theta^2 + 2\theta\beta + 2\beta^2 - (\theta+\beta)^2 = \beta^2 = 2.8^2 = 7.84.$$

The shift θ moves the mean but not the spread.

(d) By Theorem 4.12 with $g(Y) = 4 + 1.4Y$, $m_U(t) = E(e^{t(4+1.4Y)}) = e^{4t}E(e^{(1.4t)Y}) = e^{4t}m_Y(1.4t)$. Substituting,

$$m_U(t) = e^{4t} \cdot \frac{e^{4(1.4t)}}{1-2.8(1.4t)} = \frac{e^{9.6t}}{1-3.92t}, \qquad t < 0.2551.$$

This has the same form as m_Y, with shift $4 + 1.4(4) = 9.6$ and exponential mean $1.4(2.8) = 3.92$: U is again a shifted exponential.

Problem 12

seed 20260925

A property insurer records every claim above 190 thousand AZN. The size Y (thousand AZN) of a recorded claim has the Pareto distribution function

$$F(y) = \begin{cases} 0, & y < 190, \\ 1 - \left(\dfrac{190}{y}\right)^{2.5}, & y \geq 190. \end{cases}$$

Give answers to at least four significant figures.

(a) The density function for $y > 190$, as a function of y: $f(y) =$ ______________

(b) The 0.86 quantile $\phi_{0.86}$ of claim size: ______________ thousand AZN

(c) Given that a claim exceeds 570 thousand AZN, the probability that it exceeds 950 thousand AZN: ______________

(d) $E(Y) =$ ______________ thousand AZN

(e) A reinsurer pays the part of each claim above a retention of 950 thousand AZN, that is $Y - 950$ if $Y > 950$ and 0 otherwise. Its expected payment per recorded claim: ______________ thousand AZN

(a) Definition 4.3: $f(y) = dF(y)/dy$. For $y > 190$,

$$f(y) = \frac{d}{dy}\left[1 - 190^{2.5}y^{-2.5}\right] = 2.5 \cdot 190^{2.5}\, y^{-3.5} = \frac{2.5\,(190)^{2.5}}{y^{3.5}},$$

and $f(y) = 0$ for $y < 190$.

(b) Definition 4.4: solve $F(\phi) = 0.86$, i.e. $(190/\phi)^{2.5} = 0.14$:

$$\phi_{0.86} = 190\,(0.14)^{-1/2.5} = 417.1597.$$

(c) Both events are tails, $P(Y > y) = (190/y)^{2.5}$, and the second is contained in the first, so

$$P(Y > 950 \mid Y > 570) = \frac{(190/950)^{2.5}}{(190/570)^{2.5}} = \left(\frac{570}{950}\right)^{2.5} = \left(\frac{3}{5}\right)^{2.5} = 0.2789.$$

The threshold cancels. Unlike the exponential, the Pareto is **not** memoryless: the answer depends only on the ratio $950/570$, so large claims stay large.

(d) Definition 4.5:

$$E(Y) = \int_{190}^{\infty} y \cdot \frac{2.5\,190^{2.5}}{y^{3.5}}\,dy = 2.5\,190^{2.5}\left[\frac{y^{-1.5}}{-1.5}\right]_{190}^{\infty} = \frac{2.5(190)}{1.5} = 316.6667.$$

(The integral converges because $\alpha = 2.5 > 1$.)

(e) By Theorem 4.4 with $g(y) = y - 950$ on $y > 950$,

$$E[g(Y)] = \int_{950}^{\infty}(y-950)\frac{2.5\,190^{2.5}}{y^{3.5}}\,dy = 2.5\,190^{2.5}\left[\frac{950^{-1.5}}{1.5} - \frac{950 \cdot 950^{-2.5}}{2.5}\right] = \frac{190^{2.5}\,950^{-1.5}}{1.5} = 11.3294.$$

As a check, this is $P(Y > 950) \times 950/1.5 = 0.017889 \times 950/1.5$: the reinsurer is hit with probability 0.017889, and when it is hit the expected excess is $950/(\alpha - 1)$, proportional to the retention itself.

Problem 13

seed 20260925

The total cash Y (thousand AZN) withdrawn in a day from a busy ATM in central Baku has mean $\mu = 28$ and standard deviation $\sigma = 28$. Take $k = 2.88$.

Give answers to at least four significant figures, except (c), which should be given to four decimal places using Table 4.

(a) Without assuming any particular distribution, the largest lower bound that Tchebysheff's theorem gives for $P(|Y - \mu| < k\sigma)$:

____

(b) The exact value of $P(|Y - \mu| < k\sigma)$ if Y is exponential with mean 28: _______________

(c) The exact value of $P(|Y - \mu| < k\sigma)$ if instead Y were normal with the same mean and standard deviation: _______________

(d) The bank loads the ATM each morning with c thousand AZN and wants $P(Y > c) = 0.05$. Assuming Y is exponential with mean 28, $c =$ _______________

(e) A risk officer who does not trust the exponential model instead loads $c = \mu + k^*\sigma$, with k^* chosen so that Tchebysheff's theorem guarantees $P(|Y - \mu| \geq k^*\sigma) \leq 0.05$ (and hence $P(Y > c) \leq 0.05$). Find this $c =$ _______________

(a) Theorem 4.13: $P(|Y - \mu| < k\sigma) \geq 1 - 1/k^2 = 1 - 1/8.2944 = 0.8794$.

(b) For the exponential, $\mu = \sigma = \beta = 28$ (Theorem 4.10). The event is $-52.64 < Y < 108.64$; since $k > 1$ the lower limit is negative and $Y > 0$, so only the upper limit binds:

$$P(Y < (1+k)\beta) = 1 - e^{-(1+k)} = 1 - e^{-3.88} = 0.9793.$$

(c) For a normal, $P(|Y - \mu| < k\sigma) = P(-k < Z < k) = 1 - 2A(k)$, where $A(z)$ is the Table 4 tail area:
$1 - 2A(2.88) = 1 - 2(0.0020) = 0.9960$.

Both exact values exceed the bound 0.8794, as they must: Tchebysheff's theorem holds for **every** distribution, and pays for that generality by being conservative for any particular one.

(d) $P(Y > c) = e^{-c/28} = 0.05$ gives $c = 28\ln(1/0.05) = 28(2.9957) = 83.8805$.

(e) Tchebysheff gives $P(|Y - \mu| \geq k^*\sigma) \leq 1/(k^*)^2$. Setting $1/(k^*)^2 = 0.05$ gives $k^* = 1/\sqrt{0.05} = 4.4721$, and since $\{Y > \mu + k^*\sigma\} \subset \{|Y - \mu| \geq k^*\sigma\}$,

$$c = 28 + 4.4721(28) = 153.2198.$$

The distribution-free load is larger than the model-based one in (d): that is the cost of insuring against being wrong about the model.

Problem 14

seed 20260925

A telecom operator leases backbone capacity to an internet provider. For a randomly chosen month let Y_1 be the proportion of the link the provider reserves and Y_2 the proportion it actually uses; usage cannot exceed the reservation. Their joint density is

$$f(y_1, y_2) = \begin{cases} k\, y_1^1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures.

(a) The constant $k =$ ______________

(b) $P(Y_1 \le 0.85) =$ ______________

(c) $P(Y_2 \le 0.30) =$ ______________

(d) In a month in which the provider reserves 0.85 of the link, the probability that it uses at most 0.25 of it: $P(Y_2 \le 0.25 \mid Y_1 = 0.85) =$ ______________

(e) The expected proportion used, $E(Y_2) =$ ______________

(a) Theorem 5.2 requires the density to integrate to 1 over the triangle:

$$\int_0^1 \int_0^{y_1} k\, y_1^1\, dy_2\, dy_1 = \int_0^1 k\, y_1^2\, dy_1 = \frac{k}{3} = 1 \quad \Rightarrow \quad k = 3.$$

(b) The inner integral above is the marginal density (Definition 5.4): $f_1(y_1) = \int_0^{y_1} 3\, y_1^1\, dy_2 = 3\, y_1^2$ for $0 \le y_1 \le 1$. Hence $P(Y_1 \le 0.85) = 0.85^3 = 0.6141$.

(c) For fixed y_2, y_1 runs from y_2 to 1:

$$f_2(y_2) = \int_{y_2}^1 3\, y_1^1\, dy_1 = \frac{3}{2}(1 - y_2^2), \qquad 0 \le y_2 \le 1.$$

Then

$$P(Y_2 \le 0.30) = \frac{3}{2} \int_0^{0.30} (1 - y_2^2)\, dy_2 = \frac{3}{2}\left(0.30 - \frac{0.30^3}{3}\right) = 0.4365.$$

(d) Definition 5.7:

$$f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{3\, y_1^1}{3\, y_1^2} = \frac{1}{y_1}, \qquad 0 \le y_2 \le y_1.$$

Given $Y_1 = y_1$, usage is **uniform** on $[0, y_1]$, whatever the power 1. So $P(Y_2 \le 0.25 \mid Y_1 = 0.85) = 0.25/0.85 = 0.2941$.

(e) From (d), $E(Y_2 \mid Y_1) = Y_1/2$ (the mean of a uniform on $[0, Y_1]$). Theorem 5.14 then gives $E(Y_2) = E[E(Y_2 \mid Y_1)] = E(Y_1)/2$, with $E(Y_1) = \int_0^1 y_1 \cdot 3\, y_1^2\, dy_1 = 3/4 = 0.7500$:

$$E(Y_2) = \frac{0.7500}{2} = 0.3750.$$

Integrating $y_2 f_2(y_2)$ directly gives the same number with more work.

Problem 15

seed 20260925

An insurer runs a motor line and a property line with separate customers and separate call centres. Let Y_1 be the delay (days) before a motor accident is reported and Y_2 the delay before a property loss is reported, for one claim drawn at random from each line. The actuaries fit the joint density

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{18.75}\, e^{-y_1/2.5 - y_2/7.5}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures.

(a) $P(Y_1 > 2,\ Y_2 > 8) =$ ______________

(b) The probability that the motor claim is reported first, $P(Y_1 < Y_2) =$ ______________

(c) The probability that the two delays add up to at most 8 days, $P(Y_1 + Y_2 \le 8) =$ ______________

(d) $E(Y_1 Y_2) =$ ______________

(e) $V(Y_1 - Y_2) =$ ______________

The support is the rectangle (quadrant) $y_1 > 0, y_2 > 0$ and $f(y_1, y_2) = \left(\frac{1}{2.5} e^{-y_1/2.5}\right)\left(\frac{1}{7.5} e^{-y_2/7.5}\right)$ is a function of y_1 times a function of y_2, so by Theorem 5.5 the delays are independent, each exponential, with means $\beta_1 = 2.5$ and $\beta_2 = 7.5$.

(a) By independence, $P(Y_1 > 2, Y_2 > 8) = e^{-2/2.5}\, e^{-8/7.5} = 0.4493(0.3442) = 0.1546$.

(b) Integrate over $y_1 < y_2$, inner integral in y_1:

$$P(Y_1 < Y_2) = \int_0^\infty \frac{1}{\beta_2} e^{-y_2/\beta_2}\left(1 - e^{-y_2/\beta_1}\right) dy_2 = 1 - \frac{1/\beta_2}{1/\beta_2 + 1/\beta_1} = \frac{\beta_2}{\beta_1 + \beta_2} = \frac{7.5}{10.0} = 0.7500.$$

(c) Over the triangle $y_1 + y_2 \le c$ with $c = 8$:

$$P(Y_1 + Y_2 \le c) = \int_0^c \frac{1}{\beta_1} e^{-y_1/\beta_1}\left(1 - e^{-(c-y_1)/\beta_2}\right) dy_1 = 1 - \frac{\beta_1 e^{-c/\beta_1} - \beta_2 e^{-c/\beta_2}}{\beta_1 - \beta_2}.$$

With the numbers, $e^{-8/2.5} = 0.0408$ and $e^{-8/7.5} = 0.3442$, so

$$P(Y_1 + Y_2 \le 8) = 1 - \frac{2.5(0.0408) - 7.5(0.3442)}{-5.0} = 0.5042.$$

(d) Theorem 5.9: $E(Y_1 Y_2) = E(Y_1)E(Y_2) = 2.5(7.5) = 18.75$.

(e) Independence gives $\text{Cov}(Y_1, Y_2) = 0$ (Theorem 5.11), so by Theorem 5.12 with coefficients 1 and -1, $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) = 2.5^2 + 7.5^2 = 62.50$ (Theorem 4.10: the variance of an exponential is β^2). A minus sign in a linear combination does not reduce the variance; it only flips the sign of the covariance term, which here is zero.

Problem 16

seed 20260925

At a Caspian port, each import declaration is routed independently to the **green** channel (released without checks) with probability 0.73, the **yellow** channel (documentary check) with probability 0.19, or the **red** channel (physical inspection) with probability 0.08. A logistics firm files 9 declarations this week. Let Y_1, Y_2, Y_3 be the numbers routed green, yellow and red.

A documentary check costs the firm 85 AZN and a physical inspection 350 AZN, so its inspection cost is $C = 85Y_2 + 350Y_3$.

Give answers to at least four significant figures (enter decimals).

(a) $P(Y_1 = 4, Y_2 = 4, Y_3 = 1) =$ ______________

(b) The probability that none of the 9 declarations is sent to the red channel: ______________

(c) $\text{Cov}(Y_2, Y_3) =$ ______________

(d) $E(C) =$ ______________ AZN

(e) $V(C) =$ ______________ AZN^2

The 9 declarations are identical, independent trials with three outcome cells, so (Y_1, Y_2, Y_3) is multinomial (Definitions 5.11 and 5.12) with $n = 9$ and $(p_1, p_2, p_3) = (0.73, 0.19, 0.08)$.

(a) Definition 5.12:

$$P(4,4,1) = \frac{9!}{4!\,4!\,1!}(0.73)^4(0.19)^4(0.08)^1 = 630\,(0.73)^4(0.19)^4(0.08)^1 = 0.018652.$$

(b) Lumping green and yellow together, each declaration is “red” or “not red”, so Y_3 alone is binomial with $n = 9$, $p = 0.08$ (this is the marginal distribution of a multinomial count). Hence $P(Y_3 = 0) = (0.92)^9 = 0.4722$.

(c) Theorem 5.13: $\mathrm{Cov}(Y_2, Y_3) = -np_2p_3 = -9(0.19)(0.08) = -0.1368$. It is negative because the counts compete for the same 9 declarations.

(d) Theorem 5.13 gives $E(Y_i) = np_i$: $E(Y_2) = 1.71$, $E(Y_3) = 0.72$. By Theorem 5.12, $E(C) = 85(1.71) + 350(0.72) = 397.35$ AZN.

(e) Theorem 5.13: $V(Y_2) = np_2(1 - p_2) = 1.3851$ and $V(Y_3) = np_3(1 - p_3) = 0.6624$. Theorem 5.12:

$$V(C) = 85^2(1.3851) + 350^2(0.6624) + 2(85)(350)(-0.1368) = 83011.75.$$

The standard deviation of the weekly inspection bill is about 288.12 AZN. Ignoring the negative covariance would overstate the risk.

Problem 17

seed 20260925

A household-insurance portfolio is not homogeneous: each policyholder has an unobserved annual claim rate Λ, which varies across policyholders as a gamma random variable with $\alpha = 2.0$ and $\beta = 0.28$. Given $\Lambda = \lambda$, the number N of claims the policyholder makes in a year is Poisson with mean λ. Claim sizes $X_1, X_2, \ldots$ are independent of each other and of N, each with mean 1600 AZN and standard deviation 1900 AZN. The policyholder’s aggregate claim for the year is $T = X_1 + \cdots + X_N$ ($T = 0$ if $N = 0$).

Give answers to at least four significant figures.

(a) For a randomly chosen policyholder, $E(N) =$ ______________

(b) $V(N) =$ ______________

(c) $P(N = 0) =$ ______________

(d) $E(T) =$ ______________ AZN

(e) The standard deviation of T: ______________ AZN

Given Λ, $E(N \mid \Lambda) = V(N \mid \Lambda) = \Lambda$ (Theorem 3.11), and by Theorem 4.8 $E(\Lambda) = \alpha\beta = 0.5600$, $V(\Lambda) = \alpha\beta^2 = 0.156800$.

(a) Theorem 5.14: $E(N) = E[E(N \mid \Lambda)] = E(\Lambda) = 0.5600$.

(b) Theorem 5.15:

$$V(N) = E[V(N \mid \Lambda)] + V[E(N \mid \Lambda)] = E(\Lambda) + V(\Lambda) = 0.5600 + 0.156800 = 0.716800.$$

The ratio $V(N)/E(N) = 1 + \beta = 1.28 > 1$: heterogeneity between policyholders makes the counts **overdispersed** relative to a single Poisson.

(c) Condition on Λ: $P(N = 0 \mid \Lambda) = e^{-\Lambda}$, so $P(N = 0) = E(e^{-\Lambda})$, the gamma moment-generating function $m(t) = (1 - \beta t)^{-\alpha}$ evaluated at $t = -1$:

$$P(N = 0) = (1 + 0.28)^{-2.0} = (1.28)^{-2.0} = 0.6104.$$

(A homogeneous Poisson with the same mean would give $e^{-0.5600} = 0.5712$.)

(d) Given $N = n$, T is a sum of n claims, so $E(T \mid N) = N\mu_X$ and, by independence, $V(T \mid N) = N\sigma_X^2$. Theorem 5.14: $E(T) = E(N)\mu_X = 0.5600(1600) = 896.00$ AZN.

(e) Theorem 5.15:

$$V(T) = E[V(T \mid N)] + V[E(T \mid N)] = E(N)\sigma_X^2 + V(N)\mu_X^2 = 0.5600(3610000) + 0.716800(2560000) = 3856608.0,$$

so the standard deviation is $\sqrt{3856608.0} = 1963.82$ AZN. Both sources of risk appear: the size of each claim (σ_X^2) and the number of claims ($V(N)$).

Problem 18

seed 20260925

A gas utility injects gas into an underground storage site over the summer and withdraws it during the winter. The amount injected, Y_1 (million cubic metres), has a gamma distribution with $\alpha = 2$ and $\beta = 7.5$. Given $Y_1 = y_1$, the amount withdrawn over the winter, Y_2, is uniformly distributed on $(0, y_1)$ (how much is needed depends on how cold the winter is).

Give answers to at least four significant figures.

(a) In a year when 15.0 million cubic metres were injected, the probability that more than 12.0 are withdrawn: $P(Y_2 > 12.0 \mid Y_1 = 15.0) =$ ______________

(b) $E(Y_2) =$ ______________

(c) $V(Y_2) =$ ______________

(d) $\text{Cov}(Y_1, Y_2) =$ ______________

(e) The coefficient of correlation ρ between Y_1 and Y_2: ______________

Theorem 4.8: $E(Y_1) = \alpha\beta = 15.0$, $V(Y_1) = \alpha\beta^2 = 112.50$, so $E(Y_1^2) = V(Y_1) + [E(Y_1)]^2 = \alpha(\alpha+1)\beta^2 = 337.50$. Given Y_1, Theorem 4.6 for the uniform on $(0, Y_1)$ gives $E(Y_2 \mid Y_1) = Y_1/2$ and $V(Y_2 \mid Y_1) = Y_1^2/12$.

(a) With the conditional density $f(y_2 \mid 15.0) = 1/15.0$ on $(0, 15.0)$, $P(Y_2 > 12.0 \mid Y_1 = 15.0) = (15.0 - 12.0)/15.0 = 3.0/15.0 = 0.2000$.

(b) Theorem 5.14: $E(Y_2) = E[E(Y_2 \mid Y_1)] = E(Y_1)/2 = 7.50$.

(c) Theorem 5.15:

$$V(Y_2) = E[V(Y_2 \mid Y_1)] + V[E(Y_2 \mid Y_1)] = \frac{E(Y_1^2)}{12} + \frac{V(Y_1)}{4} = \frac{337.50}{12} + \frac{112.50}{4} = 28.1250 + 28.1250 = 56.2500.$$

The first term is the uncertainty about the winter for a known stock; the second is the uncertainty inherited from the summer injection.

(d) Condition on Y_1 (Theorem 5.14 applied to Y_1Y_2): $E(Y_1Y_2) = E[E(Y_1Y_2 \mid Y_1)] = E[Y_1 \cdot Y_1/2] = E(Y_1^2)/2 = 168.7500$. Then Theorem 5.10:

$$\text{Cov}(Y_1, Y_2) = 168.7500 - 15.0(7.50) = 56.2500,$$

which is exactly $V(Y_1)/2$.

(e) $\rho = \dfrac{\text{Cov}(Y_1, Y_2)}{\sigma_1\sigma_2} = \dfrac{56.2500}{\sqrt{112.50 \times 56.2500}} = 0.7071.$

A larger injection allows (and on average leads to) a larger withdrawal, so the correlation is positive, but it is far from 1 because the winter itself adds independent variation.

Problem 19

seed 20260925

A grid operator monitors a high-voltage line into Baku. Measured in months from now, let Y_1 be the time of the line's first protective trip and Y_2 the time of its second. Their joint density is

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{20.25}\, e^{-y_2/4.5}, & 0 < y_1 < y_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Give answers to at least four significant figures.

(a) The probability that there is no trip in the next 6.0 months, $P(Y_1 > 6.0) =$ ______________

(b) The probability that the line trips at least twice within 13 months, $P(Y_2 \le 13) =$ ______________

(c) $P(Y_1 \le 4 \mid Y_2 = 19) =$ ______________

(d) $\mathrm{Cov}(Y_1, Y_2) =$ ______________

(e) $V(Y_1 + Y_2) =$ ______________

Write $\beta = 4.5$, so $f(y_1, y_2) = \beta^{-2} e^{-y_2/\beta}$ on $0 < y_1 < y_2$.

(a) Definition 5.4: for fixed y_1, y_2 runs from y_1 to ∞:

$$f_1(y_1) = \int_{y_1}^{\infty} \beta^{-2} e^{-y_2/\beta}\, dy_2 = \frac{1}{\beta} e^{-y_1/\beta}, \qquad y_1 > 0,$$

an exponential with mean β. So $P(Y_1 > 6.0) = e^{-6.0/4.5} = 0.2636$.

(b) For fixed y_2, y_1 runs from 0 to y_2: $f_2(y_2) = \int_0^{y_2} \beta^{-2} e^{-y_2/\beta}\, dy_1 = y_2 e^{-y_2/\beta}/\beta^2$, a gamma density with $\alpha = 2$ (Definition 4.9). One integration by parts gives $P(Y_2 \le c) = 1 - e^{-c/\beta}(1 + c/\beta)$; with $c/\beta = 13/4.5 = 2.8889$,

$$P(Y_2 \le 13) = 1 - e^{-2.8889}(1 + 2.8889) = 0.7836.$$

(c) Definition 5.7: $f(y_1 \mid y_2) = f(y_1, y_2)/f_2(y_2) = 1/y_2$ for $0 < y_1 < y_2$. Given the second trip at month 19, the first is uniform on $(0, 19)$, so $P(Y_1 \le 4 \mid Y_2 = 19) = 4/19 = 0.2105$.

(d) $E(Y_1) = \beta = 4.5$ and $E(Y_2) = 2\beta = 9.0$ (Theorems 4.10 and 4.8). By Definition 5.9, integrating y_1 first,

$$E(Y_1 Y_2) = \int_0^{\infty} \int_0^{y_2} y_1 y_2\, \beta^{-2} e^{-y_2/\beta}\, dy_1\, dy_2 = \int_0^{\infty} \frac{y_2^3}{2\beta^2} e^{-y_2/\beta}\, dy_2 = \frac{\Gamma(4)\beta^4}{2\beta^2} = 3\beta^2 = 60.75.$$

Theorem 5.10: $\mathrm{Cov}(Y_1, Y_2) = 3\beta^2 - \beta(2\beta) = \beta^2 = 20.25$.

(e) $V(Y_1) = \beta^2$, $V(Y_2) = 2\beta^2 = 40.50$. Theorem 5.12:

$$V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\mathrm{Cov}(Y_1, Y_2) = \beta^2 + 2\beta^2 + 2\beta^2 = 5\beta^2 = 101.25.$$

The positive covariance matters: treating the two trip times as independent would give $3\beta^2$ and understate the spread. The same theorem gives $V(Y_2 - Y_1) = 2\beta^2 + \beta^2 - 2\beta^2 = \beta^2$, the variance of an exponential with mean β: the gap between trips behaves exactly like the wait for the first one, the signature of trips arriving as a Poisson process.

Problem 20

seed 20260925

Let Y_1 be next month's return (in %) on a regional equity index and Y_2 the return on the shares of a bank listed on it. The pair has a bivariate normal distribution with

$$\mu_1 = 0.7, \quad \sigma_1 = 8, \qquad \mu_2 = 2.30, \quad \sigma_2 = 10, \qquad \rho = 0.6.$$

Recall (Exercise 5.129) that, given $Y_1 = y_1$, Y_2 is normal with mean $\mu_2 + \rho \dfrac{\sigma_2}{\sigma_1}(y_1 - \mu_1)$ and variance $\sigma_2^2(1 - \rho^2)$.

Give (b) and (e) to four decimal places using Table 4; give the others to at least four significant figures.

(a) $\text{Cov}(Y_1, Y_2) =$ ______________

(b) The probability that the bank's shares lose value next month, $P(Y_2 < 0) =$ ______________

(c) A stress scenario assumes the index returns $y_1 = -14.34$. Find $E(Y_2 \mid Y_1 = -14.34) =$ ______________

(d) The conditional standard deviation of Y_2 given $Y_1 = -14.34$: ______________

(e) In the stress scenario, the probability that the bank's shares fall below -18.82 %: $P(Y_2 < -18.82 \mid Y_1 = -14.34) =$

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(a) For the bivariate normal, ρ is the correlation coefficient (Section 5.10), so $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2 = 0.6(8)(10) = 48.0$.

(b) The marginal of Y_2 is normal with mean 2.30 and standard deviation 10 (Exercise 5.128). With $z = (0 - 2.30)/10 = -0.23$, by symmetry

$$P(Y_2 < 0) = P(Z < -0.23) = P(Z > 0.23) = A(0.23) = 0.4090.$$

(c) The scenario is $y_1 - \mu_1 = -14.34 - 0.7 = -15.04$, i.e. 1.88 market standard deviations below the mean. The regression slope is $\rho\sigma_2/\sigma_1 = 0.6(10)/8 = 0.7500$, so

$$E(Y_2 \mid Y_1 = -14.34) = 2.30 + 0.7500(-15.04) = 2.30 - 6(1.88) = -8.98.$$

(d) $\sigma_2\sqrt{1-\rho^2} = 10\sqrt{0.64} = 10(0.8) = 8$. Knowing the market's return removes a fraction $\rho^2 = 0.6^2$ of the share's variance, whatever the value of y_1.

(e) Standardise with the **conditional** mean and standard deviation:

$$z = \frac{-18.82 - (-8.98)}{8} = -1.23,$$

so $P(Y_2 < -18.82 \mid Y_1 = -14.34) = P(Z < -1.23) = A(1.23) = 0.1093$.

The unconditional probability of a loss in (b) says little about what happens in a market crash; the conditional distribution shifts the whole distribution of the bank's return down by $\rho\sigma_2/\sigma_1$ per point of market fall and narrows it by the factor $\sqrt{1-\rho^2}$.

Answer key

Problem	Answers
1	(a) 0.8655, (b) 0.230597, (c) 0.1345, (d) 0.711, (e) Dependent
2	(a) 0.131357, (b) 0.776673, (c) 516.84, (d) 0.570416
3	(a) 0.453089, (b) 0.434598, (c) 1.66355, (d) 1.08741
4	(a) 0.3, (b) 0.48, (c) 1.1, (d) 0.25, (e) Not independent
5	(a) 0.39, (b) 0.36, (c) 0.4, (d) 0.2596, (e) 0.548019
6	(a) 10.375, (b) 302.4, (c) 457.06, (d) 21.379, (e) 344.68
7	(a) 0.1316, (b) 0.424012, (c) 0.141835, (d) 0.0359434, (e) 0.710291
8	(a) 0.449329, (b) 0.191208, (c) 7120, (d) 1252.2, (e) 2.22554, (f) 0.0750311
9	(a) 3.8, (b) 2.68701, (c) 0.531758, (d) 3449.2
10	(a) 20, (b) 0.333333, (c) 0.031746, (d) 0.256218, (e) 240
11	(a) e^(4*t)/(1-2.8*t), (b) 6.8, (c) 7.84, (d) e^(9.6*t)/(1-3.92*t)
12	(a) 1.24401E+06/(y^3.5), (b) 417.16, (c) 0.278855, (d) 316.667, (e) 11.3294
13	(a) 0.879437, (b) 0.979349, (c) 0.996023, (d) 83.8805, (e) 153.22
14	(a) 3, (b) 0.614125, (c) 0.4365, (d) 0.294118, (e) 0.375

15	(a) 0.154638, (b) 0.75, (c) 0.50415, (d) 18.75, (e) 62.5
16	(a) 0.0186525, (b) 0.472161, (c) -0.1368, (d) 397.35, (e) 83011.7
17	(a) 0.56, (b) 0.7168, (c) 0.610352, (d) 896, (e) 1963.82
18	(a) 0.2, (b) 7.5, (c) 56.25, (d) 56.25, (e) 0.707107
19	(a) 0.263597, (b) 0.78363, (c) 0.210526, (d) 20.25, (e) 101.25
20	(a) 48, (b) 0.409046, (c) -8.98, (d) 8, (e) 0.109348